p-BLOCKS RELATIVE TO A CHARACTER OF A NORMAL SUBGROUP II

NOELIA RIZO

ABSTRACT. Let p be a prime number, let G be a finite group, let N be a normal subgroup of G, and let θ be a G-invariant irreducible character of N. In [Riz18], we introduced a canonical partition of the set $\mathrm{Irr}(G|\theta)$ of irreducible constituents of the induced character θ^G, relative to the prime p. We call the elements of this partition the θ-blocks. In this paper, we construct a canonical basis of the complex space of class functions defined on $\{x \in G \,|\, x_p \in N\}$, which supersedes previous non-canonical constructions. This allows us to define θ-decomposition numbers in a natural way. We also prove that the elements of the partition of $\mathrm{Irr}(G|\theta)$ established by these θ-decomposition numbers are the θ-blocks.

Mathematics Classification Number: Primary 20D; Secondary 20C15
Keywords: Brauer p-blocks, Brauer's first main theorem, block orthogonality, central extensions.

1. INTRODUCTION

Let p be a prime number, and let M be a maximal ideal of the ring $\mathbf{R}$ of algebraic integers in $\mathbb{C}$ containing p. For every finite group G, and only depending on M, R. Brauer constructed a basis of the space $\mathrm{cf}(G^{p'})$ of complex class functions defined on the set $G^{p'}$ of the p-regular elements of G, namely the set $\mathrm{IBr}(G)$ of the irreducible Brauer characters of G. Suppose now that N is a fixed normal subgroup of G, and consider $G^0 = \{g \in G \,|\, g_p \in N\}$, where g_p is the p-part of $g \in G$, and let $\mathrm{cf}(G^0)$ denote the space of complex class functions defined on G^0. If χ is any complex class function defined on G, then χ^0 denotes the restriction of χ to the set G^0.

If N is a p-group, Navarro constructed in [Nav00] a canonical basis $\mathrm{IBr}(G, N)$ of $\mathrm{cf}(G^0)$ (depending only on M) satisfying that whenever $\chi \in \mathrm{Irr}(G)$, then

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G,N)} d_{\chi\varphi}\varphi$$

for some uniquely defined non-negative integers $d_{\chi\varphi}$. The *N-projective characters*

$$\Phi_\varphi = \sum_{\chi \in \mathrm{Irr}(G)} d_{\chi\varphi}\chi$$

were previously studied by B. Külshammer and G. R. Robinson in [KR87].

In the not-so-widely available [Nav12], Navarro constructed a similar basis of $\mathrm{cf}(G^0)$ for any normal subgroup N of G, where N was not necessarily a p-group. However, his methods did not allow him to prove that this basis was canonical (that is, depending only on M). In the first main result of this paper, we construct such a canonical basis.

When studying the complex space $\mathrm{cf}(G^0)$, it is well known that standard Clifford reductions (that we shall review below), allow us to fix a G-invariant character $\theta \in \mathrm{Irr}(N)$ and only consider the subspace $\mathrm{cf}(G^0|\theta)$ which is the span of $\{\chi^0 \,|\, \chi \in \mathrm{Irr}(G|\theta)\}$, where, as usual, $\mathrm{Irr}(G|\theta)$ is the set of irreducible constituents of the induced character θ^G.

Theorem A. *Suppose that p is a prime, G is a finite group, N is a normal subgroup of G, and $\theta \in \mathrm{Irr}(N)$ is G-invariant. Then there is a canonical basis $\mathrm{IBr}(G|\theta)$ of the space $\mathrm{cf}(G^0|\theta)$ (depending only on the choice of the maximal ideal M) such that whenever $\chi \in \mathrm{Irr}(G|\theta)$, then*

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi$$

for some uniquely determined non-negative integers $d_{\chi\varphi}$. If $N = 1$, then $\mathrm{IBr}(G|1) = \mathrm{IBr}(G)$.

We call the elements of this basis the θ-*Brauer characters* and the non-negative integers $d_{\chi\varphi}$ the θ-*decomposition numbers.* If N is a p'-group, it is easy to see that $\mathrm{IBr}(G|\theta)$ is exactly the set of irreducible Brauer characters of G lying over θ (as a Brauer character).

The existence of $\mathrm{IBr}(G|\theta)$ allows us to define a new natural linking in $\mathrm{Irr}(G|\theta)$: we say that $\chi, \psi \in \mathrm{Irr}(G|\theta)$ are θ-*linked* if there is $\varphi \in \mathrm{IBr}(G|\theta)$ such that $d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}$. In fact, we shall prove that the θ-blocks defined in [Riz18] can be characterized by this linking.

Theorem B. *Suppose that p is a prime, G is a finite group, N is a normal subgroup of G, and $\theta \in \mathrm{Irr}(N)$ is G-invariant. Then the connected components of the graph defined in $\mathrm{Irr}(G|\theta)$ by θ-linking are the θ-blocks.*

Once θ-Brauer characters and θ-decomposition numbers are defined, of course, many natural questions arise. However, our main interest in these new objects is to study if they can be of any help in order to understand classical problems on blocks, defect groups, Brauer characters, or decomposition numbers. For instance, as suggested by Navarro, perhaps the θ-decomposition numbers can be bounded by a function of the size of θ-defect groups (as defined in [Riz18]), and the presence of normal subgroups might facilitate reductions to simple groups.

In the last part of this paper we give analogues to classical results on blocks for the θ-blocks: a θ-version of Brauer's first main theorem and a θ-version of block-orthogonality, which we believe might have independent interest.

This paper is structured as follows: in Section 2 we recall the definition and some properties of the θ-blocks that will be used later on. In Section 3 we prove Theorem A and in Section 4 we prove Theorem B. In Section 5 we prove the θ-versions of Brauer's first main theorem and block orthogonality.

Acknowledgements. Most of this paper is part of my PhD thesis under the direction of Gabriel Navarro. I would like to take this opportunity to warmly thank him for thoroughly reading this manuscript. I would also like to thank Radha Kessar and Charles Eaton for many useful comments on this work and, in general, on my thesis.

2. Preliminaries

Let us start by recalling here the definition of θ-blocks given in [Riz18]. The notion of character triple isomorphism (see Definition 11.23 of [Isa76]), as well as the theory of projective representations, is essential in order to define and to deal with θ-blocks.

Recall that a complex projective representation of a finite group G is a map

$$\mathcal{P} : G \to \mathrm{GL}_n(\mathbb{C})$$

such that for every $x, y \in G$ there is some $\alpha(x, y) \in \mathbb{C}^\times$ satisfying

$$\mathcal{P}(x)\mathcal{P}(y) = \alpha(x, y)\mathcal{P}(xy).$$

The function $\alpha : G \times G \to \mathbb{C}^\times$ is called the factor set of $\mathcal{P}$.

If G is a finite group, $N \lhd G$, and $\theta \in \mathrm{Irr}(N)$ is G-invariant, then we say that (G, N, θ) is a character triple. The theory of character triples and their isomorphisms were developed by I. M.

Isaacs, and we refer to Chapter 11 of [Isa76] for their properties. (See also Chapter 5 of [Nav18]). Character triples have associated projective representations in the following way. If (G, N, θ) is a character triple, a projective representation $\mathcal{P}$ of G is associated with θ if

(a) $\mathcal{P}_N$ is an ordinary representation of N affording θ, and
(b) $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$ and $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ for $g \in G$ and $n \in N$.

A fundamental fact about projective representations associated to character triples is that, given a character triple, one can always find a projective representation associated to it, $\mathcal{P}$, such that its factor set α has roots of unity values (see for instance Theorem 8.2 of [Isa73] or Theorem 5.5 of [Nav18]). Using such a projective representation $\mathcal{P}$, it is possible to associate to each character triple (G, N, θ) a new finite group $\hat{G}$ constructed as follows. Let Z be a finite subgroup of the multiplicative group of the field of the complex numbers containing the values of α. As a set, $\hat{G}$ is the set of pairs $\{(g, z) \mid g \in G, z \in Z\}$, and we give to $\hat{G}$ the following product,

$$(g, z)(h, t) = (gh, \alpha(g, h)zt).$$

It turns out that $\hat{G}$ is a finite group that contains N as a normal subgroup and $Z \subseteq \mathbf{Z}(\hat{G})$. Moreover, if we define $\tau(g, z) = z\mathrm{trace}(\mathcal{P}(g))$, we have that τ is an irreducible character of $\hat{G}$ extending θ. For a more detailed explanation see Theorem 11.28 of [Isa76] or Theorem 5.6 of [Nav18]. We care to remark that $\hat{G}$ depends on the choice of Z. However, we will not emphasize this in the notation, since it is not going to matter for our purposes.

One of the keys in the construction of the θ-blocks is the following. Suppose that (G, N, θ) is a character triple, let $\hat{G}$ be as above and write $\hat{N} = N \times Z$. Then we have that $\hat{N}$ is normal in $\hat{G}$, $\hat{N}/N$ is central in $\hat{G}/N$ and there is a uniquely defined linear character $\hat{\lambda}$ of $\hat{N}/N$ such that (G, N, θ) and $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ are isomorphic character triples (see Theorem 11.28 of [Isa76] or Corollary 5.9 of [Nav18]). In particular, this means that there is a bijection

$$^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\hat{G}/N|\hat{\lambda}).$$

Since we are going to use how this bijection is explicitly constructed, we review this construction here. Let $\chi \in \mathrm{Irr}(G|\theta)$ and let $\pi : \hat{G} \to G$ be the onto group homomorphism $(g, z) \mapsto g$, which has kernel Z. Since π induces an isomorphism $\hat{G}/Z \to G$, there is a unique $\chi^\pi \in \mathrm{Irr}(\hat{G})$ such that $\chi^\pi(g, z) = \chi(g)$ for all $g \in G, z \in Z$. Since χ lies over θ notice that χ^π lies over θ. Let $\tau \in \mathrm{Irr}(\hat{G})$ be the character extending θ mentioned above. By Gallagher's Corollary 6.17 of [Isa76], there exists a unique $\chi^* \in \mathrm{Irr}(\hat{G}/N)$ such that $\chi^\pi = \chi^*\tau$. (Recall that we view the characters of H/N as characters of H that contain N in its kernel.) Now, evaluating in $(1, z)$ for $z \in Z$, we easily check that $\chi^* \in \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$.

Notation 2.1. We say that the group $\hat{G}$ constructed above is a *representation group associated with* (G, N, θ) and $\mathcal{P}$ (and Z). We usually write $G^* = \hat{G}/N$, $N^* = \hat{N}/N$ and $\theta^* = \hat{\lambda}$ and we say that the character triple (G^*, N^*, θ^*) is a *standard isomorphic character triple* for (G, N, θ) given by $\mathcal{P}$. Also, we call the bijective map $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ constructed above the *standard bijection*.

We recall now the definition of θ-blocks and θ-defect groups given in [Riz18].

Definition 2.2. Let (G, N, θ) be a character triple. Let $\hat{G}$ be a representation group for (G, N, θ) and let $\pi : \hat{G} \to G$ be the canonical homomorphism $(g, z) \mapsto g$ with kernel Z. Let $^* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(\hat{G}/N|\hat{\lambda})$ be the associated standard bijection. We say that a non-empty subset $B_\theta \subseteq \mathrm{Irr}(G|\theta)$ is a θ-*block* of G if there exists a p-block $\hat{B}$ of $\hat{G}/N$ such that

$$B_\theta^* = \{\chi^* \mid \chi \in B_\theta\} = \mathrm{Irr}(\hat{B}|\hat{\lambda})$$

where χ^* denotes the image of χ through the standard bijection. If $\hat{D}/N$ is a defect group of $\hat{B}$, then we say that $\pi(\hat{D})/N$ is a θ-*defect group* of B_θ.

Of course, as we have introduced it here, the definition of θ-blocks seems to depend on the choice of the projective representation associated with θ. The same happens with the θ-defect groups. However, one of the main results in [Riz18] is to prove that they are canonically defined (Theorem 4.3 of [Riz18]).

3. Theorem A

Suppose again that N is a normal subgroup of G and consider the normal set $G^0 = \{x \in G \,|\, x_p \in N\}$ and the complex space $\mathrm{cf}(G^0)$ of complex class functions defined on G^0. If $\delta \in \mathrm{cf}(G)$, we denote by δ^0 the restriction of δ to G^0. The space $\mathrm{cf}(G^0)$ can naturally be decomposed as a direct sum of subspaces. Indeed, for a given $\theta \in \mathrm{Irr}(N)$, we define $\mathrm{cf}(G|\theta)$ to be the $\mathbb{C}$-span of $\mathrm{Irr}(G|\theta)$, and we let

$$\mathrm{cf}(G^0|\theta) = \mathrm{cf}(G|\theta)^0 = \{\delta^0 \,|\, \delta \in \mathrm{cf}(G|\theta)\} \,.$$

Of course, $\mathrm{cf}(G^0|\theta) = \mathrm{cf}(G^0|\theta^g)$ for $g \in G$. In fact, it is not difficult to prove (see Lemma 2.1 of [Nav00]) that if Θ is a complete set of representatives of the G-action on $\mathrm{Irr}(N)$, then

$$\mathrm{cf}(G^0) = \bigoplus_{\theta \in \Theta} \mathrm{cf}(G^0|\theta) \,.$$

The strategy now is to fix $\theta \in \mathrm{Irr}(N)$ and focus on $\mathrm{cf}(G^0|\theta)$. The next natural step is to prove that if $T = G_\theta$ is the stabilizer of θ in G, then induction $\psi \mapsto \psi^G$ defines a linear isomorphism

$$\mathrm{cf}(T^0|\theta) \to \mathrm{cf}(G^0|\theta) \,.$$

This is done in Lemma 2.2 of [Nav00]. Using induction, this allows us to work with a G-invariant θ, that is, with a character triple (G, N, θ).

Suppose now that a set $\mathrm{IBr}(G)$ of irreducible Brauer characters of G is given (in other words, that we have chosen a maximal ideal M containing p in the ring of algebraic integers $\mathbf{R}$ of the complex numbers.) If N is a p-group, Navarro constructed in [Nav00] a natural basis of $\mathrm{cf}(G^0)$ depending just on $\mathrm{IBr}(G)$. Since we are using this construction explicitly, we review it here for the reader's convenience.

Suppose that $\theta \in \mathrm{Irr}(N)$ is G-invariant and N is a p-group. We define $\hat{\theta} \in \mathrm{cf}(G^0|\theta)$ as follows. If $x \in G^0$, then $x_p \in N$ and $N\langle x \rangle/N$ is a p'-group. Since N is a p-group, there is a canonical extension $\hat{\theta}_x \in \mathrm{Irr}(N\langle x \rangle)$ (Corollary 8.16 of [Isa76]). This is the unique extension of θ to $N\langle x \rangle$ whose determinantal order is a power of p. Now we define $\hat{\theta}(x) = \hat{\theta}_x(x)$. If η is any class function defined on the p-regular elements of G, we define

$$(\theta \star \eta)(x) = \hat{\theta}(x)\eta(x_{p'})$$

for $x \in G^0$. One of the main results in [Nav00] (Theorem 4.3 of [Nav00]) is that

$$\mathrm{IBr}(G|\theta) = \{\theta \star \eta \,|\, \eta \in \mathrm{IBr}(G)\}$$

is a basis of $\mathrm{cf}(G^0|\theta)$ and that if $\chi \in \mathrm{Irr}(G|\theta)$, then

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi$$

for some (uniquely defined) non-negative integers $d_{\chi\varphi}$.

What happens when N is not necessarily a p-group? This case is solved (in a non-canonical way) in [Nav12]. Suppose first that N is central. Hence $N = N_p \times N_{p'}$, where $N_p \in \mathrm{Syl}_p(N)$. Therefore we can write $\theta = \alpha \times \beta$, where $\alpha \in \mathrm{Irr}(N_p)$ and $\beta \in \mathrm{Irr}(N_{p'})$. If $\eta \in \mathrm{cf}(G) \cup \mathrm{cf}(G^{p'})$, Navarro defines

$$(\theta \star \eta)(x) = \theta(x_p)\eta(x_{p'}) = (\alpha \star \eta)(x).$$

In [Nav12] it is proved that

$$\mathrm{IBr}(G|\theta) = \{\alpha \star \eta \mid \eta \in \mathrm{IBr}(G|\beta)\}$$

is a basis of $\mathrm{cf}(G^0|\theta)$, and that if $\chi \in \mathrm{Irr}(G|\theta)$, then

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi$$

for some (uniquely defined) non-negative integers $d_{\chi\varphi}$. Notice that if $\theta \in \mathrm{IBr}(N)$ (that is, if $N = N_{p'}$), then $\alpha = 1$, $\beta = \theta$ and $\mathrm{IBr}(G|\theta)$ is exactly the set of irreducible Brauer characters of G lying over θ.

Finally, it is shown in Lemma 2.1 of [Nav12], that if (G, N, θ) and (G^*, N^*, θ^*) are isomorphic character triples then there is natural isomorphism of the vector spaces

$$* : \mathrm{cf}(G^0|\theta) \to \mathrm{cf}((G^*)^0|\theta^*)$$

such that

$$(\chi^*)^0 = (\chi^0)^*$$

for $\chi \in \mathrm{cf}(G|\theta)$. This easily shows that, if N^* is central in G^*, then the inverse image of

$$\mathrm{IBr}(G^*|\theta^*) = \{\theta_p^* \star \eta^* \mid \eta \in \mathrm{IBr}(G^*|\theta_{p'}^*)\}$$

where $\theta^* = \theta_p^* \times \theta_{p'}^*$, is a basis of $\mathrm{cf}(G^0|\theta)$. Since this basis depends on the choice of the isomorphic character triple (G^*, N^*, θ^*), we denote it by $\mathcal{B}_{(G^*, N^*, \theta^*)}$. Hence if $\chi \in \mathrm{Irr}(G|\theta)$, then

$$\chi^0 = \sum d_{\chi\varphi}\varphi$$

for some uniquely determined non-negative integers $d_{\chi\varphi}$, where φ runs over $\mathcal{B}_{(G^*, N^*, \theta^*)}$. The problem with this construction is that there is no known way of choosing a canonical (G^*, N^*, θ^*) with N^* central that is isomorphic to (G, N, θ). Our main theorem in this section solves this by proving that if we choose two standard isomorphic triples, then the corresponding bases that are obtained through this process coincide.

Assume that (G^*, N^*, θ^*) is any character triple isomorphic to (G, N, θ), with N^* central, and again write $N^* = N_p^* \times N_{p'}^*$, where $N_p^* \in \mathrm{Syl}_p(N^*)$ and hence $\theta^* = \theta_p^* \times \theta_{p'}^*$, with $\theta_p^* \in \mathrm{Irr}(N_p^*)$ and $\theta_{p'}^* \in \mathrm{Irr}(N_{p'}^*)$. By Theorem 2.4 of [Nav12], the set

$$\mathrm{IBr}(G^*|\theta^*) = \{\theta^* \star \varphi^* \mid \varphi^* \in \mathrm{IBr}(G^*|\theta_{p'}^*)\}$$

is a basis of $\mathrm{cf}((G^*)^0|\theta^*)$. Since $\mathrm{cf}(G^*|\theta^*)$ is the $\mathbb{C}$-span of $\mathrm{Irr}(G^*|\theta^*)$, for each $\varphi^* \in \mathrm{IBr}(G^*|\theta_{p'}^*)$ we can write

$$\theta^* \star \varphi^* = \sum_{\chi^* \in \mathrm{Irr}(G^*|\theta^*)} a_{\varphi^*\chi^*} (\chi^*)^{\circ}$$

for some complex numbers $a_{\varphi^*\chi^*} \in \mathbb{C}$. (We remark that these numbers are not necessarily unique, but for our purposes this is not going to matter.) Since (G, N, θ) and (G^*, N^*, θ^*) are isomorphic character triples, we know that there exists a bijection $* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ and by Lemma 2.1 of [Nav12], the map $\Psi^{\circ} \mapsto (\Psi^*)^{\circ}$ from $\mathrm{cf}(G|\theta)^{\circ} \to \mathrm{cf}(G^*|\theta^*)^{\circ}$ is an isomorphism of vector spaces. Hence the basis of $\mathrm{cf}(G|\theta)^{\circ}$ described in [Nav12] is

$$\mathcal{B}_{(G^*, N^*, \theta^*)} = \{ \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi^*\chi^*}\chi^{\circ} \mid \varphi^* \in \mathrm{IBr}(G^*|\theta_{p'}^*)\}.$$

If (G, N, θ) is a character triple and (G_1, N_1, θ_1) and (G_2, N_2, θ_2) are standard isomorphic character triples, we wish to prove that $\mathcal{B}_{(G_1,N_1,\theta_1)} = \mathcal{B}_{(G_2,N_2,\theta_2)}$. As usual, if $\chi \in \mathrm{cf}(G)$, we denote by $\chi^{p'}$ the restriction of χ to $G^{p'}$. We need the following easy observation.

Lemma 3.1. *Let* $N \lhd G$, $\theta \in \mathrm{Irr}(N)$ *linear, and* $\alpha, \beta \in \mathrm{cf}(G) \cup \mathrm{cf}(G^{p'})$*, then*

    (1) $\theta \star (\alpha + \beta) = \theta \star \alpha + \theta \star \beta$,
    (2) if $\theta \star \alpha = \theta \star \beta$ *then* $\alpha^{p'} = \beta^{p'}$.

Proof. (a) is straightforward. Now suppose that $\theta \star \alpha = \theta \star \beta$, and take $y \in G^{p'}$. Notice that $y \in G^\circ$. Since θ is linear, we have that $\alpha(y) = \theta(1)\alpha(y) = (\theta \star \alpha)(y) = (\theta \star \beta)(y) = \theta(1)\beta(y) = \beta(y)$ and (b) follows. $\square$

We will also use also the following non-trivial result of [Nav12].

Theorem 3.2. *Suppose that* $N \subseteq \mathbf{Z}(G)$*, and let* $\theta = \alpha\beta \in \mathrm{Irr}(N)$*, where* $\alpha \in \mathrm{Irr}(N)$ *has p-power order and* $\beta \in \mathrm{Irr}(N)$ *has* p'*-order. Let* $\chi \in \mathrm{cf}(G|\theta)$*, then* $\chi^\circ = \alpha \star \chi$*.*

Proof. See Theorem 2.4 of [Nav12]. $\square$

Recall that if $\alpha : \hat{G} \to G$ is a surjective group homomorphism with kernel Z and $\psi \in \mathrm{Irr}(G)$, we denote by ψ^α the unique irreducible character of $\hat{G}$ such that $\psi^\alpha(x) = \psi(\alpha(x))$ for $x \in \hat{G}$. Now, note that if $x \in \hat{G}^{p'}$, then $\alpha(x) \in G^{p'}$. Hence, if $\varphi \in \mathrm{IBr}(G)$, we denote by φ^α the unique irreducible Brauer character of $\hat{G}$ such that $\varphi^\alpha(x) = \varphi(\alpha(x))$ for $x \in \hat{G}^{p'}$. Notice that $Z \subseteq \ker(\varphi^\alpha)$.

We recall here the following result of [Riz18], which will be essential to prove Theorem A.

Lemma 3.3. *Let* (G, N, θ) *be a character triple. Let* $\mathcal{P}_1, \mathcal{P}_2$ *be projective representations of* G *associated with* θ*, with factor sets* α_1 *and* α_2*, respectively, whose values are roots of unity. Let* Z_i *be the subgroup of the multiplicative group of the field of complex numbers generated by the values of* α_i*. Let* $\hat{G}_i$ *be the representation group associated with* $\mathcal{P}_i$ *and let* $(\hat{G}_1/N, \tilde{N}_1/N, \hat{\lambda}_1)$ *and* $(\hat{G}_2/N, \tilde{N}_2/N, \hat{\lambda}_2)$ *be the standard isomorphic character triples given by* $\mathcal{P}_1$ *and* $\mathcal{P}_2$*, respectively. Let* $\hat{G} = G \times Z_1 \times Z_2$ *(as a set) and define in* $\hat{G}$ *the product*

$$(g, z_1, z_2)(h, z_1', z_2') = (gh, \alpha_1(g, h)z_1 z_1', \alpha_2(g, h)z_2 z_2').$$

Then the following hold.

    (a) $\hat{G}$ *is a finite group and* $N \times 1 \times 1$ *is a normal subgroup of* $\hat{G}$ *(which we identify with* N*).*
    (b) The maps $\rho_1 : \hat{G} \to \hat{G}_1$ *and* $\rho_2 : \hat{G} \to \hat{G}_2$ *given by* $(g, z_1, z_2) \mapsto (g, z_1)$ *and* $(g, z_1, z_2) \mapsto (g, z_2)$ *are surjective group homomorphisms with kernels* Z_2 *and* Z_1*, respectively.*
    (c) Let $\chi \in \mathrm{Irr}(G|\theta)$ *and let* $\chi_i \in \mathrm{Irr}(\hat{G}_i/N|\hat{\lambda}_i)$ *be the image of* χ *under the standard bijection. Let* $\hat{\chi}_i = \chi_i^{\rho_i} \in \mathrm{Irr}(\hat{G}/N)$*. Then there exists a linear character* $\beta \in \mathrm{Irr}(\hat{G}/N)$ *such that* $\beta\hat{\chi}_1 = \hat{\chi}_2$*.*

Proof. This is part of Theorem 4.2 of [Riz18]. $\square$

Finally we prove the following result which, as we mentioned above, will be key in order to prove Theorem A.

Theorem 3.4. *Let* (G, N, θ) *be a character triple and let* (G_1, N_1, θ_1) *and* (G_2, N_2, θ_2) *be standard isomorphic character triples. Then*

$$\mathcal{B}_{(G_1,N_1,\theta_1)} = \mathcal{B}_{(G_2,N_2,\theta_2)}.$$

Proof. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be projective representations associated with θ arising (G_1, N_1, θ_1) and (G_2, N_2, θ_2) respectively (that is, $G_i = \hat{G}_i/N$, $N_i = \hat{N}_i/N$, and $\theta_i = \hat{\lambda}_i$ in the notation of Lemma 3.3). If $\chi \in \mathrm{Irr}(G|\theta)$, we write $\chi_i \in \mathrm{Irr}(G_i|\theta_i)$ for the image of χ through the respective standard bijections. Now,

$$\mathrm{IBr}(G_1|\theta_1) = \{\theta_1 \star \varphi_1 \mid \varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})\}$$
$$= \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi_1^\circ \mid \varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})\}.$$

and

$$\mathrm{IBr}(G_2|\theta_2) = \{\theta_2 \star \varphi_2 \mid \varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})\}$$
$$= \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_2\chi_2} \chi_2^\circ \mid \varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})\}.$$

Hence,

$$\mathcal{B}_{(G_1,N_1,\theta_1)} = \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi^\circ \mid \varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})\}$$

and

$$\mathcal{B}_{(G_2,N_2,\theta_2)} = \{\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_2\chi_2} \chi^\circ \mid \varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})\}$$

Since $\mathcal{B}_{(G_1,N_1,\theta_1)}$ and $\mathcal{B}_{(G_2,N_2,\theta_2)}$ are basis of $\mathrm{cf}(G|\theta)^\circ$, we have that $|\mathcal{B}_{(G_1,N_1,\theta_1)}| = |\mathcal{B}_{(G_2,N_2,\theta_2)}|$. Therefore we just need to prove that

$$\mathcal{B}_{(G_1,N_1,\theta_1)} \subseteq \mathcal{B}_{(G_2,N_2,\theta_2)}.$$

Let

$$\psi = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi^\circ \in \mathcal{B}_{(G_1,N_1,\theta_1)}.$$

In order to prove that $\psi \in \mathcal{B}_{(G_2,N_2,\theta_2)}$ we need to show that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi_2^\circ \in \mathrm{IBr}(G_2|\theta_2).$$

In other words we need to prove that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi_2^\circ = \theta_2 \star \varphi_2$$

for some $\varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})$. By Theorem 3.2 we have that $\chi_2^\circ = \theta_2 \star \chi_2$. Then, by Lemma 3.1 (a),

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi_2^\circ = \theta_2 \star \left(\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} \chi_2\right).$$

Write

$$\varphi_2 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1} (\chi_2)^{p'}.$$

Then $\varphi_2 \in \mathrm{cf}(G_2^{p'})$ and for $g \in G_2^\circ$, we have that

$$(\theta_2 \star \varphi_2)(g) = \theta_2(g_p)\varphi_2(g_{p'}) = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_2^\circ(g).$$

To end we just need to prove that $\varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})$.

Write $\hat{G} = G \times Z_1 \times Z_2$ as in Lemma 3.3, and let $\rho_1 : \hat{G} \to \hat{G}_1$ and $\rho_2 : \hat{G} \to \hat{G}_2$ be the maps defined in Lemma 3.3(b). Write $\hat{\chi}_i = \chi_i^{\rho_i} \in \mathrm{Irr}(\hat{G})$ and let β be the linear character of $\hat{G}/N$ such that $\beta\hat{\chi}_1 = \hat{\chi}_2$ (Lemma 3.3(c)). Since β is linear and $N \subseteq \ker(\beta)$ we have that $\beta^{p'} \in \mathrm{IBr}(\hat{G}/N)$. Since $\psi \in \mathcal{B}_{(G_1,N_1,\theta_1)}$ we have that

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ \in \mathrm{IBr}(G_1|\theta_1),$$

and hence

$$\sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ = \theta_1 \star \varphi_1$$

for some $\varphi_1 \in \mathrm{IBr}(G_1|(\theta_1)_{p'})$. Now, again by Theorem 3.2 and Lemma 3.1(a) we have that

$$\theta_1 \star \varphi_1 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1^\circ = \theta_1 \star \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}\chi_1.$$

Since θ_1 is linear, by Lemma 3.1(b) we obtain that

$$\varphi_1 = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}(\chi_1)^{p'}.$$

Hence

$$\varphi_1^{\rho_1} = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}(\hat{\chi}_1)^{p'}.$$

Now, since $\beta\hat{\chi}_1 = \hat{\chi}_2$, we have that

$$\eta := \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}(\hat{\chi}_2)^{p'} = \sum_{\chi \in \mathrm{Irr}(G|\theta)} a_{\varphi_1\chi_1}(\beta\hat{\chi}_1)^{p'} = \beta^{p'}\varphi_1^{\rho_1} \in \mathrm{IBr}(\hat{G}/N).$$

Let $\hat{\rho}_2 : \hat{G}/Z_1 \to \hat{G}_2$ be the isomorphism induced by ρ_2. Since $Z_1 \subseteq \ker(\hat{\chi}_2)$, we have that $\eta = \varphi_2^{\hat{\rho}_2}$ and hence $\varphi_2 \in \mathrm{IBr}(\hat{G}_2)$. Since $N \subseteq \ker(\varphi_2)$, we obtain $\varphi_2 \in \mathrm{IBr}(G_2|(\theta_2)_{p'})$ as desired. □

As a consequence we obtain Theorem A.

Corollary 3.5. *Suppose that p is a prime, G is a finite group, N is a normal subgroup of G, and $\theta \in \mathrm{Irr}(N)$ is G-invariant. Then there is a canonical basis $\mathrm{IBr}(G|\theta)$ of the space $\mathrm{cf}(G^0|\theta)$ (depending only on the choice of the maximal ideal M) such that whenever $\chi \in \mathrm{Irr}(G|\theta)$, then*

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G|\theta)} d_{\chi\varphi}\varphi$$

for some uniquely determined non-negative integers $d_{\chi\varphi}$. If $N = 1$, then $\mathrm{IBr}(G|1) = \mathrm{IBr}(G)$.

Proof. Let $\mathrm{IBr}(G|\theta) = \mathcal{B}_{(G^*,N^*,\theta^*)}$, where (G^*, N^*, θ^*) is a standard isomorphic character triple. In Corollary 2.5 of [Nav12] it is proved that $\mathrm{IBr}(G|\theta)$ is a $\mathbb{C}$-basis of the space $\mathrm{cf}(G|\theta)^\circ$. By Theorem 3.4 we know that this basis is independent of the choice of isomorphic character triples and hence it is canonical, once we have fixed M. □

4. Theorem B

As we said in the Introduction, we call the elements of the basis $\mathrm{IBr}(G|\theta)$ the θ-Brauer characters. If $\chi \in \mathrm{Irr}(G|\theta)$ and $\varphi \in \mathrm{IBr}(G|\theta)$, we denote the coefficient of φ in χ° by $d_{\chi\varphi}$. Recall that we call the numbers $d_{\chi\varphi}$ the θ-decomposition numbers. Note that if N is a p-group these θ-decomposition numbers are the same that Navarro gives in [Nav00].

Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. We say that χ and ψ are θ-*linked* if there exists $\varphi \in \mathrm{IBr}(G|\theta)$ such that

$$d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi}.$$

The connected components of the graph define by θ-linking define a partition in $\mathrm{Irr}(G|\theta)$. We call the elements of this partition the *blocks defined by θ-decomposition numbers.* We shall prove that these blocks are, in fact, the θ-blocks.

If χ and ψ are irreducible characters of G, we say that χ and ψ are *linked* if there exists $\varphi \in \mathrm{IBr}(G)$ such that

$$d_{\chi\varphi} \neq 0 \neq d_{\psi\varphi},$$

where $d_{\chi\varphi}$ and $d_{\psi\varphi}$ are the classical decomposition numbers.

Lemma 4.1. *Let (G, N, θ) be a character triple with $N \subseteq \mathbf{Z}(G)$. Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. Then χ and ψ are linked if and only if they are θ-linked.*

Proof. Write $N = N_p \times N_{p'}$, with $N_p \in \mathrm{Syl}_p(N)$, and $\theta = \theta_p \times \theta_{p'}$, with $\theta_p \in \mathrm{Irr}(N_p)$ and $\theta_{p'} \in \mathrm{Irr}(N_{p'})$. By Theorem 3.2 we have that $\chi^\circ = \theta \star \chi = \theta \star \chi^{p'}$. Since $\chi \in \mathrm{Irr}(G|\theta_{p'})$, it is clear that all the Brauer irreducible constituents of $\chi^{p'}$ lie over $\theta_{p'}$. Now, using Lemma 3.1 we have that

$$\chi^{p'} = \sum_{\varphi \in \mathrm{IBr}(G|\theta_{p'})} d_{\chi\varphi}\varphi,$$

if and only if

$$\chi^\circ = \sum_{\varphi \in \mathrm{IBr}(G|\theta_{p'})} d_{\chi\varphi}(\theta \star \varphi).$$

$\square$

Note that from Lemma 4.1 we deduce that in the case that N is central, the θ-decomposition numbers and the classical decomposition numbers coincide. This agrees with the results obtained by J. Zeng in [Zen03].

The key to prove that the blocks defined by θ-decomposition numbers and the θ-blocks coincide is the following result of [Riz18].

Theorem 4.2. *Suppose that $N \subseteq \mathbf{Z}(G)$ and let $\theta \in \mathrm{Irr}(N)$. Let B be a Brauer p-block of G such that $\mathrm{Irr}(B|\theta)$ is not empty. Then the matrix $D_{B,\theta} = (d_{\chi\varphi})$, where $\chi \in \mathrm{Irr}(B|\theta)$, $\varphi \in \mathrm{IBr}(B)$ and $d_{\chi\varphi}$ are the classical decomposition numbers, is not of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

for any ordering of the rows and columns.

Proof. See Theorem 6.2 of [Riz18]. $\square$

Using Lemma 4.1 and Theorem 4.2 we easily obtain the following.

Theorem 4.3. *Let (G, N, θ) be a character triple with $N \subseteq \mathbf{Z}(G)$. Then the blocks defined by θ-decomposition numbers are exactly the sets $\mathrm{Irr}(B|\theta)$ where B runs over the p-blocks of G.*

Proof. Let B_θ be a block defined by θ-decomposition numbers. By Lemma 4.1 we know that $B_\theta \subseteq \mathrm{Irr}(B|\theta)$ for some p-block B. We prove now that $\mathrm{Irr}(B|\theta) \subseteq B_\theta$.

Let $D = (d_{\chi\varphi})$ be the decomposition matrix of G and write $D_{B,\theta}$ for the submatrix of D whose rows and columns are indexed by elements in $\mathrm{Irr}(B|\theta)$ and $\mathrm{IBr}(B)$ respectively. By Theorem 4.2, we know that $D_{B,\theta}$ is not of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$$

for any ordering of the rows and columns. Hence if $\chi, \psi \in \mathrm{Irr}(B|\theta)$ there exists $\chi = \chi_1, \chi_2, \dots, \chi_k = \psi$ and $\varphi_1, \varphi_2, \dots, \varphi_{k-1}$ with $\chi_i \in \mathrm{Irr}(B|\theta)$ and $\varphi_i \in \mathrm{IBr}(B)$ such that

$$d_{\chi_i\varphi_i} \neq 0 \neq d_{\chi_{i+1}\varphi_i}.$$

By Lemma 4.1, the θ-decomposition numbers are the classical decomposition numbers. This completes the proof. $\qquad\square$

The following is Theorem B.

Theorem 4.4. *Let (G, N, θ) be a character triple. The blocks defined by θ-decomposition numbers are exactly the θ-blocks of G.*

Proof. Let (G^*, N^*, θ^*) be a standard isomorphic character triple and let $* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ be the standard bijection. Let $\chi, \psi \in \mathrm{Irr}(G|\theta)$. By Lemma 2.1 of [Nav12] we have that the map $\Xi^\circ \mapsto (\Xi^*)^\circ$ from $\mathrm{cf}(G|\theta)^\circ \to \mathrm{cf}(G^*|\theta^*)^\circ$ is an isomorphism of vector spaces. Therefore, $\chi, \psi \in \mathrm{Irr}(G|\theta)$ lie in the same block defined by θ-decomposition numbers if and only if χ^*, ψ^* lie in the same block defined by θ^*-decomposition numbers. Since $N^* \subseteq \mathbf{Z}(G^*)$, using Theorem 4.3 we have that χ^*, ψ^* lie in the same block defined by θ^*-decomposition numbers if and only if χ^*, ψ^* lie in the same p-block of G^*, that is, if and only if χ and ψ lie in the same θ-block. $\qquad\square$

5. More results on θ-blocks

If $\mathrm{Bl}(G|D)$ denotes the set of p-blocks of G having defect group D, Brauer's first main theorem asserts that there is a natural bijection

$$\mathrm{Bl}(G|D) \to \mathrm{Bl}(\mathbf{N}_G(D)|D),$$

where if $B \mapsto b$, then b is known as the Brauer correspondent of B. We prove an analogue of this result for θ-blocks.

If P/N is a p-subgroup of G/N, then we will denote by $\mathrm{Bl}_\theta(G|P/N)$ the set of θ-blocks of G having P/N as a θ-defect group. If χ is a character of G, we denote by $\mathrm{bl}(\chi)$ the p-block of G containing χ.

Theorem 5.1. *Let (G, N, θ) be a character triple and let P/N be a p-subgroup of G/N. Then there exists a natural bijection*

$$\mathrm{Bl}_\theta(G|P/N) \to \mathrm{Bl}_\theta(\mathbf{N}_G(P)|P/N).$$

Proof. Let (G^*, N^*, θ^*) be a standard isomorphic character triple and let $* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ be the standard bijection. Let $\mathcal{P}$ be the projective representation of G associated with θ affording the representation group $\hat{G}$ such that $G^* = \hat{G}/N$ and $N^* = \hat{N}/N$ (see Notation 2.1). Let $\pi : \hat{G} \to G$ be the canonical onto group homomorphism $(g, z) \mapsto g$. Recall that if B_θ is a θ-block of G having defect group P/N, then there exists a block of G^*, B^*, such that $B_\theta^* = \mathrm{Irr}(B^*|\theta^*)$ and there exists $P^* = \hat{P}/N$, a defect group of B^*, such that $P/N = \pi(\hat{P})/N$.

Since $\mathbf{N}_{G^*}(P^*) \leqslant \mathbf{N}_{G^*}(P^*N^*)$, it is not difficult to see that block induction defines a bijection

$$\mathrm{Bl}(\mathbf{N}_{G*}(P^*N^*)|P^*) \to \mathrm{Bl}(G^*|P^*).$$

If $H^* \leqslant G^*$, write $\mathrm{Bl}(H^*|P^*, \theta^*)$ to denote the set of blocks c^* of H^* having P^* as a defect group and such that $|\mathrm{Irr}(c^*|\theta^*)| \neq 0$. We claim that induction defines a bijection

$$\mathrm{Bl}(\mathbf{N}_{G*}(P^*N^*)|P^*, \theta^*) \to \mathrm{Bl}(G^*|P^*, \theta^*).$$

Notice that we just need to prove that given $b^* \in \mathrm{Bl}(\mathbf{N}_{G*}(P^*N^*)|P^*)$, then $|\mathrm{Irr}(b^*|\theta^*)| \neq 0$ if and only if $|\mathrm{Irr}((b^*)^{G^*}|\theta^*)| \neq 0$. Let $b^* \in \mathrm{Bl}(\mathbf{N}_{G*}(P^*N^*)|P^*, \theta^*)$ and write $B^* = (b^*)^{G^*}$. Let $\psi^* \in \mathrm{Irr}(b^*|\theta^*)$. By Corollary 6.4 of [Nav98], we have that there exists $\chi^* \in \mathrm{Irr}(B^*)$ over ψ^*. Thus $\chi^* \in \mathrm{Irr}(B^*|\theta^*)$ and $|\mathrm{Irr}(B^*|\theta^*)| \neq 0$. Now let $B^* \in \mathrm{Bl}(G^*|P^*, \theta^*)$ and write $B^* = (b^*)^{G^*}$, where $b^* \in \mathrm{Bl}(\mathbf{N}_{G*}(P^*N^*)|P^*)$. Let $\psi^* \in \mathrm{Irr}(b^*)$ and let $\varphi^* \in \mathrm{Irr}(N^*)$ lying under ψ^*, so $\psi^* \in \mathrm{Irr}(b^*|\varphi^*)$. By Corollary 6.4 of [Nav98], there exists $\xi^* \in \mathrm{Irr}(B^*)$ over ψ^*, and hence over φ^*. By Theorem 9.2 of [Nav98], B^* covers $\mathrm{bl}(\varphi^*)$. Since $|\mathrm{Irr}(B^*|\theta^*)| \neq 0$, by Theorem 9.2 of [Nav98], we have that B^* also covers $\mathrm{bl}(\theta^*)$. By Corollary 9.3, we have that $\mathrm{bl}(\theta^*)$ and $\mathrm{bl}(\varphi^*)$ are G^*-conjugate. Since N^* is central, we have that $\mathrm{bl}(\theta^*) = \mathrm{bl}(\varphi^*)$ and therefore b^* covers $\mathrm{bl}(\theta^*)$. By Theorem 9.4 of [Nav98], there exists $\mu^* \in \mathrm{Irr}(b^*|\theta^*)$. This proves the claim.

Consider the character triple $(\mathbf{N}_G(P), N, \theta)$. We claim that $(\mathbf{N}_{G*}(P^*N^*), N^*, \theta^*)$ is a standard isomorphic character triple to $(\mathbf{N}_G(P), N, \theta)$. Write $\hat{H} = \mathbf{N}_{\hat{G}}(\hat{P}\hat{N})$. Since $\hat{N} \leqslant \hat{H} \leqslant \hat{G}$, we can write (as a set) $\hat{H} = \pi(\hat{H}) \times Z$. Write $H = \pi(\hat{H})$ and notice that $N \leqslant H \leqslant G$. We first show that $H \leqslant \mathbf{N}_G(P)$. Indeed, let $h \in H$. Then

$$P^h = (PN)^h = \pi(\hat{P}\hat{N})^{\pi(h,1)} = \pi(\hat{P}\hat{N}^{(h,1)}) = \pi(\hat{P}\hat{N}) = PN = P.$$

Notice that $\mathcal{P}_H$ is a projective representation of H associated with θ with factor set $\beta = \alpha_{H \times H}$. Since Z contains the values of β, we obtain that $(\hat{H}/N, \hat{N}/N, \theta^*)$ is standard isomorphic to the character triple (H, N, θ). Observe that

$$\mathbf{N}_{G*}(P^*N^*) = \hat{H}/N$$

and hence

$$\mathbf{N}_{G*}(P^*N^*)/N^* \cong H/N.$$

Therefore,

$$\mathbf{N}_G(P)/N = \mathbf{N}_{G/N}(P/N) \cong \mathbf{N}_{G*/N*}(P^*N^*/N^*) = \mathbf{N}_{G*}(P^*N^*)/N^* \cong H/N.$$

Since $H \leqslant \mathbf{N}_G(P)$, we obtain $H = \mathbf{N}_G(P)$. Therefore $(\mathbf{N}_{G*}(P^*N^*), N^*, \theta^*)$ is standard isomorphic to the character triple $(\mathbf{N}_G(P), N, \theta)$. Now, if c_θ is a θ-block of $\mathbf{N}_G(P)$ with defect group P/N and c^* is the corresponding block in $\mathbf{N}_{G*}(P^*N^*)$ arising c_θ, the map $c_\theta \mapsto c^*$ defines a bijection

$$* : \mathrm{Bl}_\theta(\mathbf{N}_G(P)|P/N) \to \mathrm{Bl}(\mathbf{N}_{G*}(P^*N^*)|P^*, \theta^*).$$

In the same way, the map $B_\theta \mapsto B^*$ defines a bijection

$$* : \mathrm{Bl}_\theta(G|P/N) \to \mathrm{Bl}(G^*|P^*, \theta^*).$$

We conclude that there exists a bijection

$$\mathrm{Bl}_\theta(\mathbf{N}_G(P)|P/N) \to \mathrm{Bl}_\theta(G|P/N).$$

$\square$

Of course, the definition of θ-blocks is related to projective representations, and hence with blocks of twisted group algebras. These have been studied before by many authors and, in particular, there is a version of Brauer's first main theorem for twisted group algebras by Conlon [Con64] and Reynolds [Rey66].

Our next goal is to prove a θ-version of the following classical result.

Theorem (Block orthogonality)**.** *Let $g, h \in G$ be such that g_p and h_p are not G-conjugate. If B is a block of G then*

$$\sum_{\chi \in \mathrm{Irr}(B)} \chi(g)\overline{\chi(h)} = 0.$$

Proof. See, for instance, Corollary 5.11 of [Nav98]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Recall that, given a p-element $x \in G$, the p-section of x is the set

$$S(x) = \{z \in G \mid z_p \text{ is conjugate in } G \text{ to } x\}.$$

The following is a result on classical blocks that we believe of independent interest.

Theorem 5.2. *Suppose that B is a p-block of G, $N \lhd G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Let $g, h \in G$. Suppose that $(Ng)_p$ and $(Nh)_p$ are not G/N-conjugate. Then*

$$\sum_{\chi \in \mathrm{Irr}(B|\theta)} \chi(g)\overline{\chi(h)} = 0.$$

Proof. Define the class function on G

$$\Omega = \sum_{\chi \in \mathrm{Irr}(G|\theta)} \overline{\chi(h)}\chi.$$

Let $x = g_p$. If $y \in \mathbf{C}_G(x)$ is p-regular, we claim that $\Omega(xy) = 0$. First, notice that Nxy and Nh are not G/N-conjugate. Indeed, suppose that there exists $Nw \in G/N$ such that $(Nxy)^{Nw} = Nh$. Since Nx is a p-element, Ny is a p'-element and they commute we have that $(Nxy)_p = Nx = (Ng)_p$. Then

$$Nh_p Nh_{p'} = Nh = (Nxy)^{Nw} = (Nx)^{Nw}(Ny)^{Nw}$$

and we have that $(Nx)^{Nw} = Nh_p$ a contradiction since $(Ng)_p$ and $(Nh)_p$ are not G/N-conjugate. Now, by Knörr's theorem (see, for instance, Theorem 5.21 of [Nav18]) we have that

$$\Omega(xy) = \sum_{\chi \in \mathrm{Irr}(G|\theta)} \overline{\chi(h)}\chi(xy) = 0.$$

Next we show that Ω vanishes on the p-section of x, $S(x)$. Indeed, let $z \in S(x)$, and let $u \in G$ be such that $z_p^u = x$. Then $z_{p'}^u$ is a p-regular element in $\mathbf{C}_G(x)$ and therefore $\Omega(z^u) = \Omega(xz_{p'}^u) = 0$. Since Ω is a class function, we obtain $\Omega(z) = 0$.

By Theorem 5.10 of [Nav98] we obtain

$$0 = \Omega_B(g) = \sum_{\chi \in \mathrm{Irr}(B)} [\Omega, \chi]\chi(g)$$

$$= \sum_{\chi \in \mathrm{Irr}(B)} [\sum_{\psi \in \mathrm{Irr}(G|\theta)} \overline{\psi(h)}\psi, \chi]\chi(g)$$

$$= \sum_{\chi \in \mathrm{Irr}(B)} \sum_{\psi \in \mathrm{Irr}(G|\theta)} \overline{\psi(h)}[\psi, \chi]\chi(g)$$

$$= \sum_{\chi \in \mathrm{Irr}(B|\theta)} \overline{\chi(h)}\chi(g),$$

as wanted. $\qquad\square$

As a consequence, we obtain the θ-version of the block orthogonality theorem.

Corollary 5.3. *Let (G, N, θ) be a character triple and suppose that B_θ is a θ-block of G. Let $g, h \in G$. Suppose that $(Ng)_p$ is not G/N-conjugate to $(Nh)_p$. Then*

$$\sum_{\chi \in \mathrm{Irr}(B_\theta)} \chi(g)\overline{\chi(h)} = 0.$$

Proof. Let (G^*, N^*, θ^*) be a standard isomorphic character triple and let $* : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(G^*|\theta^*)$ be the standard bijection. Write also $* : G/N \to G^*/N^*$ for the isomorphism induced by the canonical onto group homomorphism $\pi : \hat{G} \to G$ defined by $(x, z) \mapsto x$, and write $(Nx^*) = N^*x^*$. Recall that (see explanation above Notation 2.1)

$$\chi(x) = \chi^\pi(x, 1) = \tau(x, 1)\chi^*(x^*)$$

for all $x \in G$. Then

$$\sum_{\chi \in \mathrm{Irr}(B_\theta)} \chi(g)\overline{\chi(h)} = \sum_{\chi^* \in \mathrm{Irr}(B^*|\theta^*)} \tau(g, 1)\chi^*(g^*)\overline{\tau(h, 1)\chi^*(h^*)}$$

$$= \tau(g, 1)\overline{\tau(h, 1)} \sum_{\chi^* \in \mathrm{Irr}(B^*|\theta^*)} \chi^*(g^*)\overline{\chi^*(h^*)}$$

Since $(Ng)_p$ is not G/N-conjugate to $(Nh)_p$, we have that $(N^*g^*)_p$ is not G^*/N^*-conjugate to $(N^*h^*)_p$ and by Theorem 5.2 we have that

$$\sum_{\chi^* \in \mathrm{Irr}(B^*|\theta^*)} \chi^*(g^*)\overline{\chi^*(h^*)} = 0,$$

as wanted.

$\qquad\square$

REFERENCES

[Con64] S. B. Conlon, Twisted group algebras and their representations. J. Austral. Math. Soc. **4** (1964), 152–173.

[Isa73] I. M. Isaacs, Characters of solvable and symplectic groups. Amer. J. Math. **95** (1973), 594–635.

[Isa76] I.M. Isaacs. *Character theory of finite groups.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976. Pure and Applied Mathematics, No. 69.

[KR87] B. Külshammer, G. R. Robinson, Characters of relatively projective modules II. J. London Math. Soc. **36** (1987), 59–67.

[Nav98] G. Navarro. *Characters and blocks of finite groups,* volume 250 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1998.

[Nav00] G. Navarro, Brauer characters relative to a normal subgroup. Proc. London Math. Soc. (3) **81** (2000), no. 1, 55–71.

[Nav12] G. Navarro, Bases for class functions on finite groups, Biblioteca de la Revista Matemática Iberoamericana, Proceedings Encuentro en Teoria de Grupos y sus Aplicaciones (Zaragoza, 2011), (2012), 225–232.

[Nav18] G. Navarro, '*Character theory and the McKay conjecture*'. Cambridge Studies in Advanced Mathematics 175, Cambrige, 2018.

[Rey66] W. F. Reynolds, Block idempotents of twisted group algebras. Proc. Amer. Math. Soc. **17** (1966), 280–282.

[Riz18] N. Rizo. p-Blocks relative to a character of a normal subgroup. *J. Algebra* **514** (2018) 254–272.

[Zen03] J. Zeng, On relatively projective modules and the Külshammer-Robinson basis. Comm. Algebra, **31** (2003), 2425–2443.

Departamento de Matemáticas, Universidad del País Vaco/ Euskal Herriko Unibertsitatea

Email address: `noelia.rizo@ehu.eus`