

Groups with few p' -characters degrees in the principal block

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Abstract

Let $p \geq 5$ be a prime and let G be a finite group. We prove that G is p -solvable of p -length at most 2 if there are at most two distinct p' -character degrees in the principal p -block of G . This generalizes a theorem of Isaacs–Smith as well as a recent result of three of the present authors.

Keywords: p' -character degrees; principal block

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1 Introduction

Let G be a finite group. If all non-linear irreducible characters of G have degree divisible by a prime p , then G has a normal p -complement by a theorem of Thompson [Tho70, Theorem 1] (see also [Isa06, Corollary 12.2]). Moreover, Berkovich [Ber95, Proposition 9 and the subsequent remark] has shown that G is solvable in this situation. This result was extended in Kazarin–Berkovich [KB99] to the case where G has at most one non-linear character of p' -degree. In a recent paper [GRS], three of the present authors proved more generally that G is solvable of p -length at most 2 whenever $p \geq 5$ and $|\{\chi(1) : \chi \in \text{Irr}_{p'}(G)\}| \leq 2$ where $\text{Irr}_{p'}(G)$ is the set of irreducible characters of G of p' -degree. This has solved Problem 1 in [KB99, p. 588] and Problem 5.3 in [Nav16].

In the present paper we generalize our theorem to blocks. This is motivated by a result of Isaacs and Smith [IS76, Corollary 3] who showed that G has a normal p -complement if and only if all non-linear characters in the principal p -block of G have degree divisible by p . The following is our main theorem.

Theorem A. *Let B_0 be the principal block of a finite group G with respect to a prime $p \geq 5$. Suppose that $|\{\chi(1) : \chi \in \text{Irr}_{p'}(B_0)\}| \leq 2$. Then $G/\text{O}_{p'}(G)$ is solvable and $\text{O}^{pp'}(G) = 1$. In particular, G is p -solvable.*

As usual we define $\text{O}^{pp'}(G) := \text{O}^{p'}(\text{O}^p(G))$ and so on. It is easy to construct groups of p -length 2 satisfying the hypothesis of Theorem A (e.g. $G = (C_5^5 \rtimes C_{11}) \rtimes C_5$ with $p = 5$). In contrast to the main theorem of [GRS] we cannot conclude further that G is solvable since every p' -group satisfies the assumption of Theorem A. Furthermore, the examples given in [GRS] show that Theorem A does not extend to $p \in \{2, 3\}$. We also like to mention a conjecture by Malle and Navarro [MN11], which generalizes the result of Isaacs and Smith to arbitrary blocks. More precisely, they conjectured that a p -block B of G is nilpotent if and only if all height 0 characters in B have the same degree. We do not know if our main result admits a version for arbitrary blocks.

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The proof of Theorem A relies on the classification of finite simple groups. In the next section we reduce Theorem A to a statement about simple groups (Proposition 2.1 below), which is proved case-by-case in the following two sections. We care to remark that in the case of alternating groups, Proposition 2.1 is deduced as a consequence of a more general statement giving a lower bound for the number of (extendable) p' -character degrees in *any* block of maximal defect. This is Proposition 3.5 below, which we believe is of independent interest.

2 Reduction to simple groups

The following proposition about simple groups will be proven in the next two sections.

Proposition 2.1. *Let S be a finite non-abelian simple group of order divisible by a prime $p \geq 5$.*

(i) *If $S \neq P\Omega_8^+(q)$, then there exist $\alpha, \beta \in \text{Irr}(S)$ with the following properties:*

- $\alpha \neq 1 \neq \beta$,
- $\alpha(1)$ and $\beta(1)$ are not divisible by p ,
- for every $S \leq T \leq \text{Aut}(S)$, α extends to a character in the principal block of T ,
- β lies in the principal block of S and is P -invariant for some Sylow p -subgroup P of $\text{Aut}(S)$,
- $\beta(1) \nmid \alpha(1)$.

(ii) *If $S = P\Omega_8^+(q)$, then there exist $\alpha, \beta \in \text{Irr}(S)$ with the following properties:*

- $\alpha \neq 1 \neq \beta$,
- $\alpha(1)$ and $\beta(1)$ are not divisible by p ,
- $\alpha(1) > 2\beta(1)$,
- for every $S \leq T \leq \text{Aut}(S)$ there exist $\hat{\alpha}, \hat{\beta} \in \text{Irr}(\text{Aut}(T))$ in the principal block such that $\hat{\alpha}_S \in \{\alpha, 2\alpha\}$ and $\hat{\beta}_S \in \{\beta, 2\beta\}$.

We make use of the following results.

Lemma 2.2 (Murai [Mur94, Lemma 4.3]). *Let $N \trianglelefteq G$ be finite groups with principal p -blocks B_N and B_G respectively. Suppose that $\psi \in \text{Irr}_{p'}(B_N)$ is invariant under a Sylow p -subgroup of G . Then there exists a character $\chi \in \text{Irr}_{p'}(B_G)$ lying over ψ .*

Lemma 2.3. *Let $\chi, \psi \in \text{Irr}(B_0)$ where B_0 is the principal p -block of G . Suppose that $p \nmid \chi(1)$ and $\chi\psi \in \text{Irr}(G)$. Then $\chi\psi \in \text{Irr}(B_0)$.*

Proof. Clearly, $\bar{\psi} \in \text{Irr}(B_0)$. Hence by [Nav98, Corollary 3.25], we have

$$[\chi\psi, 1]^0 = [\chi, \bar{\psi}]^0 \neq 0.$$

The claim follows from [Nav98, Theorem 3.19]. □

Now we are in a position to reduce Theorem A to simple groups.

Theorem 2.4. *If Proposition 2.1 holds, then Theorem A holds.*

Proof. Let p , G and B_0 be as in Theorem A. Suppose first that G is p -solvable. Let $N := O_{p'}(G)$. Then, by [Nav98, Theorem 10.20], $\text{Irr}(B_0) = \text{Irr}(G/N)$. It follows from [GRS, Theorem A] that G/N is solvable and $O^{pp'pp'}(G/N) = 1$. In particular, $O^{pp'p}(G)N/N$ is a p' -group. Since N is a p' -group, this implies $O^{pp'pp'}(G) = 1$.

Thus, it suffices to show that G is p -solvable. Let N be a minimal normal subgroup of G . Since the principal block of G/N lies in B_0 , we may assume that G/N is p -solvable by induction on $|G|$. If N is a p -group or a p' -group, then we are done. Therefore, by way of contradiction, we assume that

$$N = S_1 \times \dots \times S_t$$

with isomorphic non-abelian simple groups $S := S_1 \cong \dots \cong S_t$ of order divisible by p . Since N is the unique minimal normal subgroup, $C_G(N) = 1$. Moreover, G permutes $S_1, \dots, S_t$ transitively by conjugation.

Case 1: $S \neq P\Omega_8^+(q)$.

Let $\alpha, \beta \in \text{Irr}(S)$ as in Proposition 2.1. We may regard α as a character of $SC_G(S)$, since $SC_G(S)/C_G(S) \cong S/Z(S) = S$. As such it extends to a character $\hat{\alpha}$ in the principal block of $N_G(S)$, because $N_G(S)/C_G(S) \leq \text{Aut}(S)$. Let $M := N_G(S_1) \cap \dots \cap N_G(S_t) \trianglelefteq G$. Since the principal block of $N_G(S)$ covers the principal block B_M of M , the restriction $\hat{\alpha}_M$ lies in B_M . Now by [Nav18, Corollary 10.5], the tensor induction $\psi := \hat{\alpha}^{\otimes G}$ is an irreducible character of G with p' -degree $\psi(1) = \alpha(1)^t$. Let $x_1, \dots, x_t \in G$ be representatives for the right cosets of $N_G(S)$ in G such that $S_1^{x_i} = S_i$. Then for $g \in M$ we obtain

$$\psi(g) = \prod_{i=1}^t \hat{\alpha}^{x_i}(g)$$

from [Nav18, Lemma 10.4]. In particular, $\psi_N = \alpha^{x_1} \times \dots \times \alpha^{x_t} \in \text{Irr}(N)$ and therefore $\psi_M \in \text{Irr}(M)$ as well. Since $\hat{\alpha}_M$ lies in B_M , so does $\hat{\alpha}_M^{x_i}$. Hence, by Lemma 2.3 also $\psi_M = \hat{\alpha}_M^{x_1} \dots \hat{\alpha}_M^{x_t}$ lies in B_M .

Let Q be a Sylow p -subgroup of M . Then $Q \cap S_i$ is a Sylow p -subgroup of S_i . It follows that $C_G(Q) \subseteq C_G(Q \cap S_i) \subseteq N_G(S_i)$ for $i = 1, \dots, t$ and therefore $C_G(Q) \subseteq M$. Hence, the Brauer correspondent B_M^G is defined (see [Nav98, Theorem 4.14]) and equals B_0 by Brauer's third main theorem. Every block B of G covering B_M has a defect group containing Q by [Nav98, Theorem 9.26]. Hence by [Nav98, Lemma 9.20], B is regular with respect to N and therefore $B = B_0$ by [Nav98, Theorem 9.19]. Thus, B_0 is the only block of G covering B_M . This implies $\psi \in \text{Irr}_{p'}(B_0)$. Since the trivial character in B_0 has degree 1, $d := \psi(1) = \alpha(1)^t$ is the unique non-trivial p' -character degree in B_0 by hypothesis.

Now we work with β . Let P be a Sylow p -subgroup of G such that β is invariant under $N_P(S)$. Without loss of generality, let $\{S_1, \dots, S_r\}$ be a P -orbit. Let $y_i \in P$ such that $S_1^{y_i} = S_i$ for $i = 1, \dots, r$. Then $\beta_i := \beta^{y_i}$ lies in the principal block of S_i . By Lemma 2.3, $\beta_1 \times \dots \times \beta_r$ lies in the principal block of N . Moreover, if $\beta_i^y \in \text{Irr}(S_j)$ for some $y \in P$, then $y_i y y_j^{-1} \in N_P(S)$. Since β is $N_P(S)$ -invariant, it follows that $\beta_i^y = \beta^{y_i y y_j^{-1} y_j} = \beta^{y_j} = \beta_j$. This shows that $\{\beta_1, \dots, \beta_r\}$ is P -orbit and $\beta_1 \times \dots \times \beta_r$ is P -invariant. If $r < t$, then we consider $\beta_{r+1} := \beta^{x_{r+1}} \in \text{Irr}(S_{r+1})$. By Sylow's theorem, we can assume after conjugation inside $N_G(S_{r+1})$ that β_{r+1} is $N_P(S_{r+1})$ -invariant. Now we can form the P -orbit of β_{r+1} to obtain another P -invariant character $\beta_{r+1} \times \dots \times \beta_s \in \text{Irr}(N)$ in the principal block of N . We repeat this with every P -orbit and eventually get a PN -invariant character

$$\tau := \beta_1 \times \dots \times \beta_t \in \text{Irr}(N)$$

in the principal block of N . Since $o(\tau) = 1$ and $\gcd(\tau(1), |PN/N|) = 1$, τ extends to PN (see [Isa06, Corollary 8.16]). By Lemma 2.2, there exists some $\chi \in \text{Irr}_{p'}(B_0)$ such that τ is a constituent of χ_N . Since $1 \neq \beta(1)^t = \tau(1) \mid \chi(1)$, it follows that $\chi(1) = d = \psi(1)$. But then $\beta(1)^t \mid \psi(1) = \alpha(1)^t$ and $\beta(1) \mid \alpha(1)$, a contradiction to the choice of α and β .

Case 2: $S = P\Omega_8^+(q)$.

Let $\alpha, \beta \in \text{Irr}(S)$ and $\hat{\alpha}, \hat{\beta} \in \text{Irr}(N_G(S))$ as in Proposition 2.1. Since the principal block of $N_G(S)$ covers B_M , $\hat{\alpha}_M$ is the sum of at most two irreducible characters in B_M . If $\alpha_1 \in \text{Irr}(B_M)$ is one of those summands, then $\alpha_1^{x_1} \dots \alpha_1^{x_t}$ restricts to $\alpha^{x_1} \times \dots \times \alpha^{x_t} \in \text{Irr}(N)$. Hence, by Lemma 2.3, $\alpha_1^{x_1} \dots \alpha_1^{x_t}$ lies in B_M . As in Case 1 we see that $(\hat{\alpha}^{\otimes G})_M$ is a sum of irreducible characters in B_M . Moreover, $(\hat{\alpha}^{\otimes G})_N = d(\alpha^{x_1} \times \dots \times \alpha^{x_t})$ where $d \in \{1, 2^t\}$. Since B_0 is the only block of G covering B_M , all irreducible constituents of $\hat{\alpha}^{\otimes G}$ lie in B_0 . We may choose such a constituent χ of p' -degree. Then $\chi_N = e(\alpha^{x_1} \times \dots \times \alpha^{x_t})$ for some integer $e \leq d \leq 2^t$. Similarly, we choose a constituent ψ of $\hat{\beta}^{\otimes G}$ with p' -degree. Then by Proposition 2.1 we derive the contradiction

$$\alpha(1)^t > 2^t \beta(1)^t \geq \psi(1) = \chi(1) \geq \alpha(1)^t. \quad \square$$

3 Alternating groups

This section is devoted to proving Proposition 2.1 for the alternating groups $S = \mathfrak{A}_n$ where $n \geq 5$. It is well-known that $\text{Aut}(S) \cong \mathfrak{S}_n$ is the symmetric group unless $n = 6$.

Given $n \in \mathbb{N}$ we let $\mathcal{P}(n)$ be the set of partitions of n . Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathcal{P}(n)$. Adopting the notation of [Ols93, Chapter 1] we let $\ell(\lambda) = k$ denote the number of parts of λ , and $\mathcal{Y}(\lambda)$ be the Young diagram of λ . Given a node $(i, j) \in \mathcal{Y}(\lambda)$ we denote by $h_{ij}(\lambda)$ the length of the hook corresponding to (i, j) . If $q \in \mathbb{N}$ then the q -core $C_q(\lambda)$ of λ is the partition obtained from λ by successively removing all hooks of length q (usually called q -hooks). We denote by $\mathcal{H}^q(\lambda)$ the subset of nodes of $\mathcal{Y}(\lambda)$ having associated hook-length divisible by q . A partition γ is called a q -core if $\mathcal{H}^q(\gamma) = \emptyset$.

The set $\text{Irr}(\mathfrak{S}_n)$ is naturally in bijection with $\mathcal{P}(n)$. Given $\lambda \in \mathcal{P}(n)$ we let χ^λ be the corresponding irreducible character of $\mathfrak{S}_n$. Let p be a prime and $\lambda, \mu \in \mathcal{P}(n)$. By [JK81, 6.1.21] we know that χ^λ and χ^μ lie in the same p -block of $\mathfrak{S}_n$ if and only if $C_p(\lambda) = C_p(\mu)$. If γ is a p -core partition then we denote by $B(\mathfrak{S}_n, \gamma)$ the corresponding p -block of $\mathfrak{S}_n$. We use the notation $\lambda \vdash_{p'} n$ to say that χ^λ has degree coprime to p .

The following result follows from [Mac71] and it will be extremely useful for our purposes.

Lemma 3.1. *Let p be a prime and let n be a natural number with p -adic expansion $n = \sum_{j=0}^k a_j p^j$. Let λ be a partition of n . Then $\lambda \vdash_{p'} n$ if and only if $|\mathcal{H}^{p^k}(\lambda)| = a_k$ and $C_{p^k}(\lambda) \vdash_{p'} n - a_k p^k$.*

A straightforward consequence of Lemma 3.1 is that $\text{Irr}_{p'}(B(\mathfrak{S}_n, \gamma)) \neq \emptyset$ if and only if $|\gamma| < p$.

For $\lambda \in \mathcal{P}(n)$, we denote by λ' its conjugate partition. From [JK81, 2.5.7] we know that $\psi^\lambda := (\chi^\lambda)_{\mathfrak{A}_n}$ is irreducible if and only if $\lambda \neq \lambda'$. In this case χ^λ and $\chi^{\lambda'}$ are all the extensions of ψ^λ to $\mathfrak{S}_n$. Let λ, μ be non-self-conjugate partitions of n . Then ψ^λ and ψ^μ lie in the same p -block of $\mathfrak{A}_n$ if and only if $C_p(\lambda) \in \{C_p(\mu), C_p(\mu)'\}$. It follows that also p -blocks of $\mathfrak{A}_n$ can be labeled by p -core partitions, by keeping in mind that conjugated p -cores label the same p -block. We denote by $B(n; \gamma)$ the p -block of $\mathfrak{A}_n$ labeled by γ .

In order to show that Proposition 2.1 holds for alternating groups, we introduce the following conventions.

Notation 1. Let B be a p -block of $\mathfrak{A}_n$. We let $\text{cd}_{p'}^{\text{ext}}(B)$ be the set of degrees of irreducible characters in B of degree coprime to p that extend to an irreducible character of $\mathfrak{S}_n$. Moreover, when S is a subset of $\mathcal{P}(n)$ we let $\text{cd}(S) = \{\chi^\lambda(1) \mid \lambda \in S\}$.

Observe that if B is the principal p -block of $\mathfrak{A}_n$ and ψ^λ lies in B and extends to $\mathfrak{S}_n$, then one of the two extensions of ψ^λ lies in the principal p -block of $\mathfrak{S}_n$. This is explained in [Ols90]. Even if in this article we are mainly interested in studying the principal block, in Proposition 3.5 below we are going to compute an explicit lower bound for $|\text{cd}_{p'}^{\text{ext}}(B(n, \gamma))|$, for any p -core γ such that $|\gamma| < p$.

Given $\gamma = (\gamma_1, \dots, \gamma_\ell) \vdash n$ and natural numbers x and y , we denote by $\gamma \star (x, y)$ the partition of $n + x + y$ defined by

$$\gamma \star (x, y) = (\gamma_1 + x, \gamma_2, \dots, \gamma_\ell, 1^y).$$

We start by proving a technical lemma that will be useful later in this section.

Lemma 3.2. *Let p be a prime, let $m, n, w \in \mathbb{N}$ be such that $m < p$ and $n = m + pw$. Let $\gamma \vdash m$ and let $a \in \mathbb{N}$ be such that $\lfloor \frac{w+1}{2} \rfloor + 1 \leq a \leq w$. Setting $\lambda = \gamma \star (ap, (w-a)p)$ and $\mu = \gamma \star ((a-1)p, (w-a+1)p)$, we have that $\chi^\lambda(1) < \chi^\mu(1)$.*

Proof. For $\nu \vdash n$ we let $\pi(\nu) := \prod h_{ij}(\nu)$ be the product of the hook-lengths in ν . From the hook length formula [JK81, 2.3.21] it follows that $\chi^\nu(1)\pi(\nu) = n!$. We let $h^i = h_{1i}(\gamma)$ and $h_j = h_{j1}(\gamma)$ for all $i \in \{1, \dots, \gamma_1\}$ and all $j \in \{1, \dots, \ell(\gamma)\}$. It follows that

$$\pi(\lambda) = (ap)! \cdot ((w-a)p)! \cdot \prod_{i=2}^{\gamma_1} (h^i + ap) \cdot \prod_{i=2}^{\ell(\gamma)} (h_i + (w-a)p) \cdot \widehat{\gamma} \cdot (h_{11}(\gamma) + pw),$$

where $\hat{\gamma}$ is the product of the hook lengths $h_{ij}(\gamma)$ for all $i, j \geq 2$. Similarly

$$\pi(\mu) = ((a-1)p)! \cdot ((w-a+1)p)! \cdot \prod_{i=2}^{\gamma_1} (h^i + (a-1)p) \cdot \prod_{i=2}^{\ell(\gamma)} (h_i + (w-a+1)p) \cdot \hat{\gamma} \cdot (h_{11}(\gamma) + pw).$$

It follows that $\pi(\lambda)/\pi(\mu) = A \cdot B \cdot C$, where

$$A = \prod_{i=1}^p \frac{(a-1)p + i}{(w-a)p + i}, \quad B = \prod_{i=2}^{\gamma_1} \frac{h^i + ap}{h^i + (a-1)p}, \quad \text{and} \quad C = \prod_{i=2}^{\ell(\gamma)} \frac{h_i + (w-a)p}{h_i + (w-a+1)p}.$$

We remark that we always regard empty products as equal to 1. We observe that $B \geq 1$. Since $a-1 \geq w-a+1$ by hypothesis, it is clear that $A > 1$. Hence, if $\ell(\gamma) = 1$ then $C = 1$ and clearly $A \cdot B \cdot C > 1$. Suppose that $\ell(\gamma) \geq 2$. Then observe that $p > |\gamma| > h_2 > h_3 > \dots > h_{\ell(\gamma)} \geq 1$. Hence for all $i \in \{2, \dots, \ell(\gamma)\}$ we have that $\frac{(a-1)p+h_i}{(w-a)p+h_i}$ is one of the factors appearing in A . Moreover

$$\frac{(a-1)p+h_i}{(w-a)p+h_i} \cdot \frac{h_i + (w-a)p}{h_i + (w-a+1)p} \geq 1,$$

since $a-1 \geq w-a+1$. We conclude that $A \cdot B \cdot C \geq A \cdot C > 1$ and therefore that $\chi^\lambda(1) < \chi^\mu(1)$. $\square$

Definition 3.3. Let p be a prime and $n = wp + m$, for some $m < p$. Let γ be a p -core partition of m . We let $H(n; \gamma)$ be the subset of $\mathcal{P}(n)$ defined by

$$H(n; \gamma) = \{\lambda \vdash_{p'} n \mid C_p(\lambda) = \gamma, \lambda = \gamma \star (a, n-m-a)\}.$$

We also set $\Omega(n; \gamma) = \{\lambda \in H(n; \gamma) \mid \lambda_1 > (\lambda')_1\}$.

Lemma 3.4. Let $n = \sum_{i=0}^k a_i p^i$ be the p -adic expansion of n , with $a_k \neq 0$. If $\gamma \vdash a_0$, then

$$|\text{cd}(\Omega(n; \gamma))| = |\Omega(n; \gamma)| \geq \lfloor \frac{a_k + 1}{2} \rfloor \cdot \prod_{i=1}^{k-1} (a_i + 1).$$

Proof. Let $\lambda = \gamma \star (x, n - a_0 - x)$, for some $0 \leq x \leq n - a_0$. Let $x = \sum_{i=0}^t b_i p^i$ be the p -adic expansion of x . By definition of $H(n; \gamma)$ we have that $\lambda \in H(n; \gamma)$ if and only if $\lambda \vdash_{p'} n$ and $C_p(\lambda) = \gamma$. In turn, this is equivalent to have that p divides x (and $n - a_0 - x$) so that $C_p(\lambda) = \gamma$ and by Lemma 3.1 to have that $b_0 = 0$ and $0 \leq b_i \leq a_i$ for all $i \geq 1$. It follows that $|\mathcal{H}(n; \gamma)| = \prod_{i=1}^k (a_i + 1)$. Moreover, if $b_k \geq \lfloor a_k/2 \rfloor + 1$, then certainly $\lambda_1 > (\lambda')_1$ and therefore $\lambda \in \Omega(n; \gamma)$. It follows that

$$|\Omega(n; \gamma)| \geq \lfloor \frac{a_k + 1}{2} \rfloor \cdot \prod_{i=1}^{k-1} (a_i + 1).$$

We conclude by observing that Lemma 3.2 implies that given $\lambda, \mu \in \Omega(n; \gamma)$ we have that $\chi^\lambda(1) \neq \chi^\mu(1)$ and hence that $|\text{cd}(\Omega(n; \gamma))| = |\Omega(n; \gamma)|$. $\square$

Given $\lambda \in \Omega(n; \gamma)$ we have that χ^λ lies in $B(\mathfrak{S}_n; \gamma)$ and that $(\chi^\lambda)_{\mathfrak{A}_n}$ is irreducible and lies in $B(n; \gamma)$. As explained in Notation 1 above, $\text{cd}_{p'}^{\text{ext}}(B(n; \gamma))$ denotes the set of degrees of irreducible characters of $B(n; \gamma)$ of degree coprime to p that extend to $B(\mathfrak{S}_n; \gamma)$.

In the following proposition we are able to establish a lower bound for the number of extendable p' -character degrees lying in any given p -block of $\mathfrak{A}_n$. We believe this statement of independent interest from the topic of this article.

Proposition 3.5. Let $n = \sum_{i=0}^k a_i p^i$ be the p -adic expansion of n , with $a_k \neq 0$. Let $\gamma \vdash a_0$, then

$$|\text{cd}_{p'}^{\text{ext}}(B(n; \gamma))| \geq \lfloor \frac{a_k + 1}{2} \rfloor \cdot \prod_{i=1}^{k-1} (a_i + 1).$$

Proof. By definition, for every partition $\lambda \in \Omega(n; \gamma)$ we have that $(\chi^\lambda)_{\mathfrak{A}_n}$ is a p' -degree character that lies in $B(n; \gamma)$ and extends to χ^λ in $B(\mathfrak{S}_n; \gamma)$. The statement now follows from Lemma 3.4. $\square$

Proposition 3.6. *Let $n \geq 5$ be a natural number and $p > 3$ be a prime. Then Proposition 2.1 holds for $\mathfrak{A}_n$. In particular if $n \geq 7$ then $|\text{cd}_{p'}^{\text{ext}}(B_0(\mathfrak{A}_n))| \geq 3$.*

Proof. Direct verification proves that Proposition 2.1 holds for $\mathfrak{A}_5$ and $\mathfrak{A}_6$. Suppose that $n \geq 7$ and that $n = a_0 + \sum_{i=1}^k a_i p^{n_i}$ is the p -adic expansion of n , with $a_i \neq 0$ for all $i \geq 1$ and with $n_1 < n_2 < \dots < n_k$. Since p is odd, for $P \in \text{Syl}_p(\mathfrak{S}_n)$ we have that $P \leq \mathfrak{A}_n$ and hence that all irreducible characters in $B_0(\mathfrak{A}_n)$ are P -invariant. Thus we just need to show that $|\text{cd}_{p'}^{\text{ext}}(B_0(\mathfrak{A}_n))| \geq 3$. From Proposition 3.5, we deduce that $|\text{cd}_{p'}^{\text{ext}}(B_0(\mathfrak{A}_n))| \geq 3$, whenever $k \geq 3$. Suppose that $k \leq 2$. If $a_0 \leq 1$ then $\text{Irr}_{p'}(B_0(\mathfrak{A}_n)) = \text{Irr}_{p'}(\mathfrak{A}_n)$ and the statement follows from [GRS, Proposition 3.5]. Hence we can assume that $a_0 \geq 2$ and consider $\lambda, \mu \in \mathcal{P}(n)$ to be defined as follows.

$$\lambda = (a_0, 1^{n-a_0}), \quad \text{and} \quad \mu = (a_0, 2, 1^{n-a_0-2}).$$

It is clear that both $(\chi^\lambda)_{\mathfrak{A}_n}$ and $(\chi^\mu)_{\mathfrak{A}_n}$ lie in the principal p -block of $\mathfrak{A}_n$ and extend to the principal p -block of $\mathfrak{S}_n$, to χ^λ and χ^μ respectively. Moreover λ and μ label characters of degree coprime to p by Lemma 3.1. Using the hook-length formula we verify that $1 = \chi^{(n)}(1) < \chi^\lambda(1) < \chi^\mu(1)$. The proof is complete. $\square$

4 Sporadic groups and groups of Lie type

Proposition 4.1. *Proposition 2.1 holds for all sporadic simple groups S and the Tits group ${}^2F_4(2)'$.*

Proof. Recall that $|\text{Aut}(S) : S| \leq 2$. Hence, we may take a p' -character $\hat{\alpha}$ in the principal block of $\text{Aut}(S)$ such that $\alpha := \hat{\alpha}_S \neq 1$ is irreducible. For β we can choose any non-trivial p' -character in the principal block of S . Now it can be checked with GAP [GAP18] that there are choices such that $\beta(1) \nmid \alpha(1)$. $\square$

Now we consider simple groups S of Lie type, by which we mean groups of the form $G/Z(G)$, where $G = \mathbf{G}^F$ is the set of fixed points of a simple simply connected algebraic group under a Steinberg morphism F . In the case where $Z(G)$ is trivial, we define $\tilde{G} = G$, and otherwise we let $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be a regular embedding, as in [CE04, Section 15], so that $Z(\tilde{\mathbf{G}})$ is connected, $[\tilde{\mathbf{G}}, \tilde{\mathbf{G}}] = [\mathbf{G}, \mathbf{G}]$, and G is normal in $\tilde{G} := \tilde{\mathbf{G}}^F$. We write $\tilde{S}$ for the group $\tilde{G}/Z(\tilde{G})$, so $\text{Aut}(S)$ may be viewed as generated by $\tilde{S}$ and graph and field automorphisms.

Recall that the set $\text{Irr}(\tilde{G})$ can be partitioned into so-called Lusztig series $\mathcal{E}(\tilde{G}, s)$, where s is a semisimple element of the dual group $\tilde{G}^*$, up to conjugacy. Each series $\mathcal{E}(\tilde{G}, s)$ has a unique character of degree $[\tilde{G}^* : C_{\tilde{G}^*}(s)]_{q'}$, where F_q is the field over which G is defined, called a semisimple character. Further, the characters in the series $\mathcal{E}(\tilde{G}, 1)$ are called unipotent characters, and for a prime p , any p -block containing a unipotent character is called a unipotent block.

When $\mathbf{G}$ is type A_{n-1} (that is, in the case of linear and unitary groups), we will use the notation $PSL_n^\epsilon(q)$ to denote $PSL_n(q)$ for $\epsilon = 1$ and $PSU_n(q)$ for $\epsilon = -1$, and similar for $GL_n^\epsilon(q)$ and $SL_n^\epsilon(q)$. Similarly, $A_{n-1}^\epsilon(q)$ will denote the untwisted case $A_{n-1}(q)$ when $\epsilon = 1$ and the twisted case ${}^2A_{n-1}(q)$ when $\epsilon = -1$. We also remark that the group $P\Omega_{2n}^+(q)$ corresponds to $D_n(q)$ and $P\Omega_{2n}^-(q)$ corresponds to ${}^2D_n(q)$.

The following result settles Proposition 2.1 for most simple groups in defining characteristic.

Proposition 4.2. *Let S be a simple group of Lie type defined over F_q , where q is a power of $p > 3$ not in the following list: $PSL_2(q)$, $PSL_3(q)$, or $PSp_4(q)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(B_0(S))$ such that $\chi_1(1) \neq \chi_2(1)$ and:*

- If $S \neq P\Omega_8^+(q)$, then for every $S \leq T \leq \text{Aut}(S)$, each of χ_1 and χ_2 extend to a character in the principal p -block of T .
- If $S = P\Omega_8^+(q)$, then $\chi_1(1) > 2\chi_2(1)$ and for every $S \leq T \leq \text{Aut}(S)$, for $i = 1, 2$, there exist $\hat{\chi}_i$ in the principal p -block of T such that $\hat{\chi}_i|_S \in \{\chi_i, 2\chi_i\}$.

Proof. In the proof of [GRS, Proposition 4.3], it is shown that there exist two characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(\tilde{S})$ that restrict irreducibly to S , extend to characters of $\text{Aut}(S)$, have different degrees, and are obtained from characters of $\tilde{G}$ trivial on $Z(\tilde{G})$. Now, since $\text{Irr}_{p'}(\tilde{G}) = \text{Irr}_{p'}(B_0(\tilde{G}))$ (using, for example, [CE04, 6.18, 6.14, 6.15]) and using [CE04, Lemma 17.2], we see that in fact these characters are members of the principal block of $\tilde{S}$, and their restrictions are members of the principal block of S .

Now, let $S \leq T \leq \text{Aut}(S)$. Then for $i = 1, 2$, $\chi_i|_{T \cap \tilde{S}}$ is in the principal block of $T \cap \tilde{S}$, since $B_0(\tilde{S})$ covers a unique block of $T \cap \tilde{S}$. Note that by [Nav98, Theorem 9.4], there must be a character of $B_0(T)$ lying above $\chi_i|_{T \cap \tilde{S}}$. If $S \neq P\Omega_8^+(q)$, we have $\text{Aut}(S)/\tilde{S}$ is abelian, and hence every character of T lying above $\chi_i|_{T \cap \tilde{S}}$ is an extension, completing the proof in this case.

If $S = P\Omega_8^+(q)$, then $\text{Aut}(S)/\tilde{S}$ is of the form $\mathfrak{S}_3 \times C$, where C is cyclic. Then the character $\hat{\chi}_i$ in $B_0(T)$ lying above $\chi_i|_{T \cap \tilde{S}}$ must be such that $\hat{\chi}_i|_S \in \{\chi_i, 2\chi_i\}$, as desired. Switching the roles of the semisimple elements s_1 and s_2 constructed in [GRS, Proposition 4.3], we further see that the characters have been constructed to satisfy $\chi_1(1) > 2\chi_2(1)$, since the centralizers of s_1 and s_2 have types $A_1 \times T_1$ and $A_1^3 \times T_2$ with T_1 and T_2 appropriate tori, and $2|C_{G^*}(s_1)|_{p'} < |C_{G^*}(s_2)|_{p'}$. $\square$

The following handles the exceptional cases left by Proposition 4.2.

Proposition 4.3. *Let S be one of $PSL_2(q)$, $PSL_3^\epsilon(q)$, or $PSp_4(q)$, where q is a power of a prime $p > 3$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(B_0(S))$ such that $\chi_2(1) \nmid \chi_1(1)$; χ_2 is invariant under a Sylow p -subgroup of $\text{Aut}(S)$; and for every $S \leq T \leq \text{Aut}(S)$, χ_1 extends to a character in the principal p -block of T .*

Proof. In this case, characters χ_1 and χ_2 are constructed in the proof of [GRS, Lemma 4.4] that satisfy all of the needed properties, except possibly the property that for every $S \leq T \leq \text{Aut}(S)$, χ_1 extends to a character in the principal block of T . However, χ_1 is again constructed from a character of $\tilde{G}$ trivial on $Z(\tilde{G})$ that restricts irreducibly to G . Hence since again $\text{Aut}(S)/\tilde{S}$ is abelian, the proof is complete arguing as in the second paragraph of Proposition 4.2. $\square$

For the remainder of the section, we consider the case of non-defining characteristic. That is, we assume $p > 3$ is a prime and S is a simple group of Lie type defined in characteristic different than p .

Proposition 4.4. *Let $p > 3$ be a prime and let S be a simple group of Lie type defined over $\mathbb{F}_q$, where q is a power of a prime different than p and S is not in the following list: $PSL_2(q)$, $PSL_3^\epsilon(q)$ with $p \mid (q + \epsilon)$, ${}^2B_2(2^{2a+1})$ with $p \mid (2^{2a+1} - 1)$, or ${}^2G_2(3^{2a+1})$ with $p \mid (3^{2a+1} - 1)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(B_0(S))$ such that $\chi_1(1) \neq \chi_2(1)$ and:*

- If $S \neq P\Omega_8^+(q)$, then for every $S \leq T \leq \text{Aut}(S)$, each of χ_1 and χ_2 extend to a character in the principal p -block of T .
- If $S = P\Omega_8^+(q)$, then $\chi_1(1) > 2\chi_2(1)$ and for every $S \leq T \leq \text{Aut}(S)$, for $i = 1, 2$, there exist $\hat{\chi}_i$ in the principal p -block of T such that $\hat{\chi}_i|_S \in \{\chi_i, 2\chi_i\}$.

Proof. We adapt our proof of [GRS, Proposition 4.5], ensuring that we may choose unipotent characters of p' -degree satisfying the principal block conditions required here. That is, we will exhibit unipotent characters of $\tilde{G}$ with different degree (and in the case of $P\Omega_8^+(q)$, satisfying $\chi_1(1) > 2\chi_2(1)$) that are contained in $\text{Irr}_{p'}(B_0(\tilde{G}))$, which as unipotent characters must be trivial on $Z(\tilde{G})$ and restrict irreducibly to G . Then the restriction lies in $B_0(G)$, since $B_0(\tilde{G})$ covers a unique block of G , and by [CE04, Lemma 17.2], the resulting characters of $\tilde{S}$ and $S = G/Z(G)$ also lie in the principal blocks. By [Mal08, Theorems 2.4 and 2.5], every unipotent character extends to its inertia group in $\text{Aut}(S)$, and except for some specifically stated exceptions, the inertia group is all of $\text{Aut}(S)$. Then arguing as in Proposition 4.2, the required properties will hold for each $S \leq T \leq \text{Aut}(S)$.

To see that the unipotent characters exhibited are indeed of p' -degree, it will often be useful to recall that $q^s - 1 = \prod_{m \mid s} \Phi_m$ and note that $p \mid \Phi_m$ if and only if $m = dp^i$ for some non-negative integer i , where Φ_m denotes the m -th cyclotomic polynomial in q and d is the order of q modulo p . Further, p^2 divides Φ_m only if $m = d$. (This is [Mal07, Lemma 5.2].)

First, we consider groups of exceptional type. If S is one of ${}^2G_2(3^{2a+1})$ or ${}^2B_2(2^{2a+1})$ but not one of the exceptions of the statement, then the unipotent characters mentioned in the proof of [GRS, Proposition 4.5] work here, since by [H90, Proposition 3.2], respectively [B79, Section 2], there is a unique unipotent block of maximal defect. If S is ${}^2F_4(2^{2a+1})$, then by [Mal90, Bemerkung 1], there is again a unique unipotent block of maximal defect unless $p \mid (2^{2a+1} - 1)$, in which case the principal block contains the Steinberg character and two more unipotent characters of p' -degree. Hence we are also done in this case. If $S = {}^3D_4(q)$, then there is either a unique unipotent block of maximal defect or the principal block contains the Steinberg character and one other unipotent character of p' -degree, using [DM87, Propositions 5.6 and 5.8], so we are similarly finished in this case.

Now let S be one of $G_2(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, or $E_8(q)$. Let d be the order of q modulo p . Using [E00, Theorem A], we have the unipotent blocks of $\tilde{G}$ are indexed by conjugacy classes of pairs (L, λ) for L a d -split Levi subgroup and λ a d -cuspidal unipotent character. In particular, the characters in the d -Harish-Chandra series indexed by such an (L, λ) all lie in the same block of $\tilde{G}$. Further, [Mal07, Corollary 6.6] then yields that if a unipotent character in the series indexed by (L, λ) has p' -degree, then L is the centralizer of a Sylow d -torus. Now, using this and [BMM93, Theorem 5.1], we see that either such an L is a maximal torus (yielding a unique block containing unipotent characters of p' degree, and hence we are done using [GRS, Proposition 4.5] again) or we may use the decompositions in [BMM93, Table 2] to see there are at least two non-trivial unipotent characters in the principal block with different degrees relatively prime to p . (For an example of the argument in the latter situation, consider $E_8(q)$ in the case $d = 7$. Then Line 58 of [BMM93, Table 2] shows that the trivial character and the unipotent characters $\phi_{8,91}$ and $\phi_{400,7}$ in the notation of [Ca85, Section 13.9], which have degree $q^{91}\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}$ and $\frac{1}{2}q^6\Phi_2^4\Phi_5^2\Phi_6^2\Phi_8\Phi_{10}^2\Phi_{14}\Phi_{15}\Phi_{18}\Phi_{20}\Phi_{24}\Phi_{30}$, respectively, lie in the same d -Harish-Chandra series, and hence the same block. Since $p \mid \Phi_7$ and $p \neq 2$, we see these two non-trivial character degrees are p' and distinct.)

We are left to consider the classical groups, in which case the unipotent characters of $\tilde{G}$ are parametrized by certain partitions or symbols. By a symbol of rank n , we mean a pair of partitions $(\begin{smallmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \cdots & \mu_b \end{smallmatrix}) = (\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix})$, where $\lambda_1 < \lambda_2 < \cdots < \lambda_a$, $\mu_1 < \mu_2 < \cdots < \mu_b$, λ_1 and μ_1 are not both 0, and $n = \sum_i \lambda_i + \sum_j \mu_j - \lfloor (\frac{a+b-1}{2})^2 \rfloor$. (The symbol $(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix})$ is equivalent to $(\begin{smallmatrix} \mu \\ \lambda \end{smallmatrix})$, and if λ_1 and μ_1 are both 0, the symbol is equivalent to $(\begin{smallmatrix} \lambda_2-1 & \cdots & \lambda_a-1 \\ \mu_2-1 & \cdots & \mu_b-1 \end{smallmatrix})$.) The defect of a symbol is $|b - a|$. Given an integer e , an e -hook is a pair of non-negative integers (x, y) with $y - x = e$, $x \notin \lambda$ (resp. μ), and $y \in \lambda$ (resp. μ). The e -core of a symbol is obtained by successively removing e -hooks, which means replacing y by x in λ (resp. μ) and then replacing the result with an equivalent symbol satisfying that λ_1 and μ_1 are not both 0. An e -cohook is defined similarly, except that $x \notin \lambda$ and $y \in \mu$ (or $x \notin \mu$ and $y \in \lambda$), and the e -cocore is obtained by removing e -cohooks, which means removing y from μ and adding x to λ (resp. removing y from λ and adding x to μ), and again replacing the result with an equivalent symbol satisfying that λ_1 and μ_1 are not both 0.

Tables 1 through 4 describe two unipotent characters for each classical type satisfying the properties described in the first paragraph and not in the list of exceptions of [Mal08, Theorem 2.5]. For each type, we include a brief discussion, but we remark that a more complete description of the degrees of such characters and the partitions and symbols can be found in [Ca85, Section 13.8], and a more complete discussion of their distribution into blocks may be found in [FS82, FS89]. We will include the details for type A_{n-1} in this respect, and note that the other types have similar arguments.

Types A_{n-1} and ${}^2A_{n-1}$. Here $\tilde{G} = GL_n^\epsilon(q)$. In this case, let e be the order of ϵq modulo p . The unipotent characters are in bijection with partitions of n , and two such characters are in the same block if and only if they have the same e -core. In particular, the trivial character is given by the partition (n) , which has e -core (r) , where $0 \leq r < e$ is the remainder when $n := me + r$ is divided by e . Table 1 lists the desired unipotent characters in this case when $n \geq 4$. Indeed, consider the case $\epsilon = 1$. The partitions listed have e -core (r) , and hence the corresponding characters are in the principal block and it suffices to show that they have p' -degree. Since $p \nmid q$, we need only consider the part of the degree relatively prime to q , which are listed following [Ca85, Section 13.8]. Consider line 1 of Table 1 in this case. Since $me + k$ is not divisible by e for $1 \leq k < e$, we see $(q^{me+k} - 1)$ contains no factors of the form Φ_{ep^i} . Hence we see $(q^{me+1} - 1) \cdots (q^n - 1)$ is not divisible by p . Similarly, if $r + 1 \neq e$, then $(q^{me-r-1} - 1)$ is not divisible by p . If $r + 1 = e$, then $(q^{me-e} - 1)/(q^e - 1)$ is divisible by p only if $p \mid (m - 1)$, so that $(q^{me-e} - 1)$ has factors of the form Φ_{ep^i} with $i \geq 1$. Hence the character listed in line 1 has p' -degree, given the stated conditions, and similar for lines 4 and 5. Line 3 refers to the Steinberg character, which is certainly of p' -degree. So, consider the characters in lines 2 and 6, of degree $\prod_{i=1}^e \frac{q^{n-i}-1}{q^i-1}$,

Table 1: Some unipotent characters in $\text{Irr}_{p'}(B_0(S))$ for type $A_{n-1}^\epsilon(q)$ with $n \geq 4$ and $p \nmid q$

	Additional condition on $n = me + r, r < e$	Partition	$\chi(1)_{q'}$
χ_1	$e \neq r + 1$ or $p \nmid (m - 1)$	$(r + 1, me - 1)$	$\frac{(q^{me+1} - \epsilon^{me+1})(q^{me+2} - \epsilon^{me+2}) \dots (q^n - \epsilon^n)(q^{me-r-1} - \epsilon^{me-r-1})}{(q - \epsilon)(q^2 - 1) \dots (q^{r+1} - \epsilon^{r+1})}$
	$e = r + 1$ and $p \mid (m - 1)$	$(1^{r+1}, me - 1)$	$\prod_{i=1}^e \frac{(q^{n-i} - \epsilon^{n-i})}{(q^i - \epsilon^i)}$
χ_2	$r < 2$	(1^n)	1
	$r \geq 2, e \neq r + 1$ or $m \geq 2$, and $e \neq r + 2$ or $p \nmid (m - 1)$	$(1, r + 1, me - 2)$	$\frac{(q^{me+1} - \epsilon^{me+1})(q^{me+2} - \epsilon^{me+2}) \dots (q^n - \epsilon^n)(q^{me-r-2} - \epsilon^{me-r-2})(q^{me-1} - \epsilon^{me-1})}{(q^{r+2} - \epsilon^{r+2})(q - \epsilon)(q^2 - 1) \dots (q^r - \epsilon^r)}$
	$r \geq 2, m = 1, e = r + 1$	$(1, e - 1, e - 1)$	$\frac{(q^{e+2} - \epsilon^{e+2})(q^{e+3} - \epsilon^{e+3}) \dots (q^n - \epsilon^n)}{(q - \epsilon)(q^2 - 1) \dots (q^{e-2} - \epsilon^{e-2})}$
	$r \geq 2, e = r + 2, p \mid (m - 1)$	$(1^{r+2}, me - 2)$	$\prod_{i=1}^e \frac{(q^{n-i} - \epsilon^{n-i})}{(q^i - \epsilon^i)}$

with $p \mid (m - 1)$. If p divides $\prod_{i=1}^e \frac{q^{n-i} - 1}{q^i - 1}$, then $p \mid (q^{n-r} - 1)/(q^e - 1) = (q^{me} - 1)/(q^e - 1)$, and hence $p \mid m$, a contradiction. The argument is similar in the case $\epsilon = -1$.

Finally, if $n = 3$ and $p \nmid (q + \epsilon)$, then note that $e = 1$ or 3 , $r < 2$, and the characters listed in Table 1 still satisfy our conditions. (In this case, the two characters are the Steinberg character and the unipotent character of degree $q(q + \epsilon)$.)

Types B_n and C_n . Here the unipotent characters of $\tilde{G}$ are in bijection with symbols of rank n and odd defect. In this case, we let e be the order of q^2 modulo p . Then two symbols are in the same block if and only if they have the same e -core, respectively e -cocore, if $p \mid q^e - 1$, respectively $p \mid q^e + 1$. The trivial character is represented by the symbol $\left(\begin{smallmatrix} n \\ \emptyset \end{smallmatrix}\right)$, which has e -core and e -cocore $\left(\begin{smallmatrix} r \\ \emptyset \end{smallmatrix}\right)$, where $0 \leq r < e$ is the remainder when $n := me + r$ is divided by e . Table 2 lists the desired unipotent characters in this case, as long as $n \neq 2$ or q is not an odd power of 2. When $n = 2$ and q is an odd power of 2, we have $e = 1$ or 2 , so we may still take the Steinberg character for χ_2 , but the the characters listed for χ_1 are not necessarily fixed by the exceptional graph automorphism (see [Mal08, Theorem 2.5(c)]). Here we may instead take the character indexed by $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 1 \end{smallmatrix}\right)$ of degree $(q + 1)^2/2$ when $p \mid (q - 1)$, and otherwise we use the character of degree $(q - 1)^2/2$ indexed by $\left(\begin{smallmatrix} 0 & 1 & 2 \\ & \emptyset & \emptyset \end{smallmatrix}\right)$.

Type D_n and 2D_n . In this case the unipotent characters of $\tilde{G}$ are in bijection with symbols of rank n and defect 0 (mod 4), respectively 2 (mod 4) in case D_n , respectively 2D_n . Again, let e be the order of q^2 modulo p , and let $n = me + r$ where $0 \leq r < e$ is the remainder when n is divided by e . The block distribution is described the same way as for types B_n and C_n .

For type $D_n(q)$, the trivial character is represented by the symbol $\left(\begin{smallmatrix} n \\ \emptyset \end{smallmatrix}\right)$, which has e -core $\left(\begin{smallmatrix} r \\ \emptyset \end{smallmatrix}\right)$ if $e \nmid n$ and $\left(\begin{smallmatrix} \emptyset \\ \emptyset \end{smallmatrix}\right)$ if $e \mid n$. It has e -cocore $\left(\begin{smallmatrix} r \\ \emptyset \end{smallmatrix}\right)$ if m is even and $e \nmid n$; $\left(\begin{smallmatrix} 0 & r \\ & \emptyset \end{smallmatrix}\right)$ if m is odd and $e \nmid n$; $\left(\begin{smallmatrix} \emptyset \\ \emptyset \end{smallmatrix}\right)$ if m is even and $e \mid n$; and $\left(\begin{smallmatrix} e \\ \emptyset \end{smallmatrix}\right)$ if m is odd and $e \mid n$. Table 3 lists the desired unipotent characters as long as $n \geq 5$. (In some cases, more than two characters are listed.) We remark that if $n = e$, then it must be that $p \mid (q^e - 1)$.

For $D_4(q) = P\Omega_8^+(q)$, note that $1 \leq e \leq 3$ and that $p \mid (q^2 + 1)$ when $e = 2$. Then the Steinberg character of degree q^{12} , labeled by $\left(\begin{smallmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{smallmatrix}\right)$ may be taken for χ_1 . For χ_2 , we take the character labeled by $\left(\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right)$, of degree $q(q^2 + 1)^2$ when $e = 1$ or 3 , and $\left(\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right)$ of degree $\frac{1}{2}q^3(q + 1)^3(q^3 + 1)$ when $e = 2$. In either case, we have $\chi_1(1) > 2\chi_2(1)$.

For type ${}^2D_n(q)$, the trivial character is represented by the symbol $\left(\begin{smallmatrix} 0 & n \\ & \emptyset \end{smallmatrix}\right)$, which has e -core $\left(\begin{smallmatrix} 0 & r \\ & \emptyset \end{smallmatrix}\right)$ when $e \nmid n$ and $\left(\begin{smallmatrix} 0 & e \\ & \emptyset \end{smallmatrix}\right)$ if $e \mid n$. The e -cocore is $\left(\begin{smallmatrix} 0 & r \\ & \emptyset \end{smallmatrix}\right)$ if $e \nmid n$ and m is even, $\left(\begin{smallmatrix} r \\ \emptyset \end{smallmatrix}\right)$ if $e \nmid n$ and m is odd, $\left(\begin{smallmatrix} e \\ \emptyset \end{smallmatrix}\right)$ if $e \mid n$ and m is even, and $\left(\begin{smallmatrix} \emptyset \\ \emptyset \end{smallmatrix}\right)$ if $e \mid n$ and m is odd. Table 4 lists the desired unipotent characters in this case. $\square$

Proposition 4.5. *Let $p > 3$ be a prime and let q be a power of a prime different than p . Let S be one of $PSL_2(q)$, $PSL_3^\epsilon(q)$ with $p \mid (q + \epsilon)$, ${}^2B_2(2^{2a+1})$ with $p \mid (2^{2a+1} - 1)$, or ${}^2G_2(3^{2a+1})$ with $p \mid (3^{2a+1} - 1)$. Then there exist two non-trivial characters $\chi_1, \chi_2 \in \text{Irr}_{p'}(B_0(S))$ such that $\chi_2(1) \nmid \chi_1(1)$; χ_2 is invariant under a Sylow p -subgroup of $\text{Aut}(S)$; and for every $S \leq T \leq \text{Aut}(S)$, χ_1 extends to a character in the principal p -block of T .*

Proof. First suppose S is $PSL_2(q)$ or $PSL_3^\epsilon(q)$ with $p \mid (q + \epsilon)$. In these cases the order of q modulo p is 1 or 2, and there is a unique unipotent block of maximal defect, so χ_1 may still be taken to be the Steinberg

Table 2: Some unipotent characters in $\text{Irr}_{p'}(B_0(S))$ for types $B_n(q)$, $C_n(q)$ with $n \geq 2$, $p \nmid q$, $(n, q) \neq (2, 2^{2a+1})$

	Conditions on $n = me + r$, $r < e$	Symbol	$\chi(1)_{q'}$ (possibly excluding factors of $\frac{1}{2}$)
χ_1	$p \mid (q^e - 1)$	$\begin{pmatrix} 0 & r+1 \\ me \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-r-1} + 1)(q^{me+1})(q^{r+1} - 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r+1)} - 1)}$
	$p \mid (q^e + 1), m \text{ odd}$	$\begin{pmatrix} 0 & me \\ r+1 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-r-1} + 1)(q^{me-1})(q^{r+1} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r+1)} - 1)}$
	$p \mid (q^e + 1), m \text{ even}$	$\begin{pmatrix} r+1 & me \\ 0 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-r-1} - 1)(q^{me+1})(q^{r+1} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r+1)} - 1)}$
χ_2	$e \mid n$	$\begin{pmatrix} 0 & 1 & \dots & n-1 & n \\ 1 & \dots & n-1 & n \end{pmatrix}$	1
	$p \mid (q^e - 1), e \nmid n, e \neq r+1 \text{ or } p \nmid (m-1)$	$\begin{pmatrix} r+1 & me \\ 0 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-r-1} - 1)(q^{me+1})(q^{r+1} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r+1)} - 1)}$
	$p \mid (q^e - 1), e \nmid n, e = r+1, p \mid (m-1)$	$\begin{pmatrix} 0 & me \\ e \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-e+1})(q^{me-1})(q^e + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2e} - 1)}$
	$p \mid (q^e + 1), e \nmid n, m \text{ odd}$	$\begin{pmatrix} 0 & 1 & me \\ 1 & r+2 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-r-2} + 1)(q^{2(me-1)} - 1)(q^{me-1})}{(q^2 - 1)^2(q^4 - 1) \dots (q^{2r-1})(q^{r+2} - 1)}$
	$p \mid (q^e + 1), e \nmid n, m \text{ even}$	$\begin{pmatrix} 1 & r+2 & me \\ 0 & 1 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2n} - 1)(q^{me-r-2} - 1)(q^{2(me-1)} - 1)(q^{me+1})}{(q^2 - 1)^2(q^2 - 1)(q^4 - 1) \dots (q^{2r-1})(q^{r+2} - 1)}$

 Table 3: Some unipotent characters in $\text{Irr}_{p'}(B_0(S))$ for type $D_n(q)$ with $n \geq 5$, $p \nmid q$

	Conditions on $n = me + r$, $r < e$	Symbol	$\chi(1)_{q'}$ (possibly excluding factors of $\frac{1}{2}$)
χ_1	$e \nmid n$	$\begin{pmatrix} me \\ r \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n - 1)(q^{me-r} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2r-1})}$
	$p \mid (q^e - 1), e \mid n$; or $p \mid (q^e + 1), e \mid n, m \text{ even}$; or $e \mid (n-1)$	$\begin{pmatrix} 0 & 1 & \dots & n-1 \\ 1 & \dots & n \end{pmatrix}$	1
	$p \mid (q^e + 1), 1 \neq e \mid n, m \text{ odd}$	$\begin{pmatrix} 1 & n-e \\ 0 & e+1 \end{pmatrix}$	$\frac{(q^{2(n-e+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n - 1)(q^{n-2e-1} + 1)(q^{n-e+1})(q^{n-e-1} - 1)}{(q-1)(q^2 - 1)(q^4 - 1) \dots (q^{2(e-1)} - 1)(q^e - 1)(q^{e+1} + 1)}$
χ_2	$p \mid (q^e - 1), e \nmid n$	$\begin{pmatrix} 1 & me \\ 0 & r+1 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n - 1)(q^{me-r-1} + 1)(q^{me+1})(q^{me-1} - 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r-1)} - 1)(q^{r-1})(q^{r+1} + 1)(q-1)}$
	$e \mid n, e \neq 1 \text{ or } p \nmid (n-1)$, with $p \mid (q^e - 1)$ or $m \text{ even}$	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	$\frac{(q^{2(n-1)} - 1)}{(q^2 - 1)}$
	$p \mid (q^2 - 1), p \mid (n-1)$	$\begin{pmatrix} n-1 \\ 1 \end{pmatrix}$	$\frac{(q^n - 1)(q^{n-2} + 1)}{q^2 - 1}$
	$p \mid (q^e + 1), e \nmid n, m \text{ even}$	$\begin{pmatrix} r+1 & me \\ 0 & 1 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n - 1)(q^{me-r-1} - 1)(q^{me+1})(q^{me-1} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r-1)} - 1)(q^{r-1})(q^{r+1} - 1)(q+1)}$
	$p \mid (q^e + 1), e \nmid n, m \text{ odd}$	$\begin{pmatrix} 0 & me \\ 1 & r+1 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n - 1)(q^{me-r-1} + 1)(q^{me-1})(q^{me-1} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r-1)} - 1)(q^{r+1})(q^{r+1} - 1)(q-1)}$
	$p \mid (q^e + 1), e \mid n, m \text{ odd}, p \nmid (m-2)$	$\begin{pmatrix} n-e \\ e \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^{me-1} - 1)(q^{me-1})(q^{me-2e+1})}{(q^2 - 1)(q^4 - 1) \dots (q^{2e} - 1)}$
	$p \mid (q^e + 1), e \mid n, m \text{ odd}, p \nmid (m-1)$	$\begin{pmatrix} 1 & n-e+1 \\ 0 & e \end{pmatrix}$	$\frac{(q^{2(n-e+2)} - 1) \dots (q^{2(n-1)} - 1)(q^n - 1)(q^{n-2e+1} + 1)(q^{n-e+1} + 1)(q^{n-e-1})}{(q-1)(q^2 - 1)(q^4 - 1) \dots (q^{2(e-2)} - 1)(q^{e-1} - 1)(q^e + 1)}$

 Table 4: Some unipotent characters in $\text{Irr}_{p'}(B_0(S))$ for type ${}^2D_n(q)$ with $n \geq 4$, $p \nmid q$

	Conditions on $n = me + r$, $r < e$	Symbol	$\chi(1)_{q'}$ (possibly excluding factors of $\frac{1}{2}$)
χ_1	$e \nmid n$	$\begin{pmatrix} r & me \\ 0 & \emptyset \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{me-r} - 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2r-1})}$
	$p \mid (q^e + 1), 1 \neq e \mid n, m \text{ odd}$ or $p \mid (q^2 - 1), p \nmid (n-1)$	$\begin{pmatrix} 0 & 1 & n \\ 1 \end{pmatrix}$	$\frac{(q^{2(n-1)} - 1)}{(q^2 - 1)}$
	$p \mid (q^2 - 1), p \mid (n-1)$	$\begin{pmatrix} 1 & n-1 \\ \emptyset \end{pmatrix}$	$\frac{(q^n + 1)(q^{n-2} - 1)}{q^2 - 1}$
	$p \mid (q^e + 1), 1 \neq e \mid n, m \text{ even}$	$\begin{pmatrix} 0 & 1 & n-e \\ e+1 \end{pmatrix}$	$\frac{(q^{2(n-e+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{n-2e-1} + 1)(q^{n-e-1})(q^{n-e-1} - 1)}{(q+1)(q^2 - 1)(q^4 - 1) \dots (q^{2(e-1)} - 1)(q^e - 1)(q^{e+1} - 1)}$
	$p \mid (q^e - 1), 1 \neq e \mid n$	$\begin{pmatrix} 1 & e+1 & n-e \\ 0 \end{pmatrix}$	$\frac{(q^{2(n-e+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{n-2e-1} - 1)(q^{n-e+1})(q^{n-e-1} - 1)}{(q-1)(q^2 - 1)(q^4 - 1) \dots (q^{2(e-1)} - 1)(q^e + 1)(q^{e+1} - 1)}$
χ_2	$p \mid (q^e - 1), e \nmid n$	$\begin{pmatrix} 0 & 1 & r+1 \\ me \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{me-r-1} + 1)(q^{me+1})(q^{me-1} + 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r-1)} - 1)(q^{r+1})(q^{r+1} + 1)(q+1)}$
	$p \mid (q^e + 1), e \nmid n, m \text{ even}$	$\begin{pmatrix} 1 & r+1 & me \\ 0 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{me-r-1} - 1)(q^{me+1})(q^{me-1} - 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r-1)} - 1)(q^{r+1})(q^{r+1} - 1)(q-1)}$
	$p \mid (q^e + 1), e \nmid n, m \text{ odd}$	$\begin{pmatrix} 0 & 1 & me \\ r+1 \end{pmatrix}$	$\frac{(q^{2(me+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{me-r-1} + 1)(q^{me-1})(q^{me-1} - 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2(r-1)} - 1)(q^{r-1})(q^{r+1} - 1)(q+1)}$
	$p \mid (q^e + 1), e \mid n, m \text{ odd}$ or $e \mid (n-1)$	$\begin{pmatrix} 0 & 1 & \dots & n \\ 1 & \dots & n-1 \end{pmatrix}$	1
	$p \mid (q^e + 1), 1 \neq e \mid n, m \text{ even}$	$\begin{pmatrix} 0 & e+1 & n-e \\ 1 \end{pmatrix}$	$\frac{(q^{2(n-e+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{n-2e-1} - 1)(q^{n-e-1})(q^{n-e-1} + 1)}{(q-1)(q^2 - 1)(q^4 - 1) \dots (q^{2(e-1)} - 1)(q^e - 1)(q^{e+1} + 1)}$
	$p \mid (q^e - 1), 1 \neq e \mid n, p \nmid (m-2)$	$\begin{pmatrix} e & n-e \\ \emptyset \end{pmatrix}$	$\frac{(q^{2(n-e+1)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{n-2e} - 1)}{(q^2 - 1)(q^4 - 1) \dots (q^{2e} - 1)}$
	$p \mid (q^e - 1), 1 \neq e \mid n, p \nmid (m-1)$	$\begin{pmatrix} 0 & 1 & n-e+1 \\ e \end{pmatrix}$	$\frac{(q^{2(n-e+2)} - 1) \dots (q^{2(n-1)} - 1)(q^n + 1)(q^{n-2e+1} + 1)(q^{n-e+1} - 1)(q^{n-e-1})}{(q+1)(q^2 - 1)(q^4 - 1) \dots (q^{2(e-2)} - 1)(q^{e-1} - 1)(q^e - 1)}$

character. Let δ be an element of order p in F_{q^2} . Write $q = \ell^a$, for some prime $\ell \neq p$, and write $a = p^b c$ with $p \nmid c$. Then $p \mid \ell^{2c} - 1$ since the order of ℓ modulo p divides $2a$, and hence $2c$. Then δ is either fixed or inverted by F_ℓ^c , where F_ℓ is the generating field automorphism. In particular, since the semisimple classes of $\tilde{G}^* \cong GL_2(q)$, resp. $GL_3^*(q)$, are determined by their eigenvalues, this means that a semisimple element s of $\tilde{G}^*$ with eigenvalues $\{\delta, \delta^{-1}\}$, respectively $\{\delta, \delta^{-1}, 1\}$ is conjugate to its image under F_ℓ^c . Thus the corresponding semisimple character of $\tilde{G}$ is fixed by F_ℓ^c , and hence a Sylow p -subgroup of $\text{Aut}(S)$. Further, s satisfies (1)-(2) of [GRS, Section 4.1.1], that is, s is a member of $[\tilde{G}^*, \tilde{G}^*] \cong SL_2(q)$, resp. $SL_3^*(q)$, and is not conjugate to sz for any $z \in Z(\tilde{G}^*)$, since $|\delta| \geq 5$. Then this character is irreducible on G and trivial on the center. Further, it has degree $(q - \eta)$, where $\eta \in \{\pm 1\}$ is such that $p \mid (q + \eta)$ for $PSL_2(q)$, and degree $q^3 - \epsilon$ for $PSL_3^*(q)$ with $p \mid (q + \epsilon)$. Since s is a p -element, the character lies in a unipotent block, and hence $B_0(\tilde{G})$, using [CE04, Theorem 9.12]. Then as in the first paragraph of Proposition 4.4, the corresponding character of S lies in the principal block. It also has non-trivial degree prime to q , which therefore does not divide the degree of the Steinberg character. Hence this character satisfies our conditions.

Now let S be ${}^2B_2(q^2)$ with $q^2 = 2^{2a+1}$ and $p \mid (q^2 - 1)$ and write $2a + 1 = p^b c$ with $p \nmid c$. Let s be such that γ^s has order $p \mid (2^c - 1)$, where γ has order $q^2 - 1$. Then using [B79, Section 2] and arguing as in the case above, we see that a slight modification of the characters used in [GRS, Lemma 4.8] works here: we may take χ_1 to be the Steinberg character and χ_2 to be the character $\chi_5(s)$ in CHEVIE notation.

Finally, let S be ${}^2G_2(q^2)$ with $q^2 = 3^{2a+1}$ and $p \mid (q^2 - 1)$. Again write $2a + 1 = p^b c$ with $p \nmid c$. Using [H90, Proposition 3.2], there is a unique unipotent block of maximal defect, so we may take χ_1 again to be the Steinberg character. For χ_2 , it follows from [H90, Proposition 4.1] and arguments as above that we may take the character $\chi_{11}(s)$ in CHEVIE notation, where now s is such that γ^s has order $p \mid (3^c - 1)$ and γ has order $q^2 - 1$. $\square$

Proposition 2.1 now follows from Propositions 3.6 and 4.1 through 4.5, completing the proof of Theorem A.

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