A REAL VERSION OF THOMPSON'S THEOREM ON DEGREES

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Abstract. We extend Thompson's theorem by taking into account real-valued irreducible characters. In particular, we prove that if G is a finite group generated by its 2-elements and there is an odd prime p dividing the order of the group G, then there is a non-linear real-valued irreducible character of G of degree coprime to p.

1. Introduction

There are two fundamental results on primes and character degrees of finite groups. On the one hand, if G is a finite group, p is a prime and $P \in \mathrm{Syl}_p(G)$, then the Itô-Michler theorem asserts that p does not divide $\chi(1)$ for all $\chi \in \mathrm{Irr}(G)$ if and only if $P \trianglelefteq G$ (and P is abelian). Dually, Thompson's theorem states that if p divides the character degrees of all non-linear irreducible characters of G, then G has a normal p-complement, that is, a normal subgroup of G of order coprime to p and index a power of p.

A successful and non-trivial line of research has been to fix $F \subseteq \mathbb{C}$ a field, consider the (in general, much smaller) subset $\mathrm{Irr}_F(G)$ of the irreducible characters of G that have values in the field F, and check to what extent the Itô-Michler or the Thompson's theorem admit F-versions. For instance, for $p = 2$, the real version of the Itô-Michler theorem is done in [1], the real version for Thompson's theorem is done in [5], and the deeper rational version of this theorem is achieved in [6]. For odd primes p, the real version of Itô-Michler is achieved in [3] and [8]: if p does not divide the real character degrees of G, then $\mathbf{O}^{2'}(G)$ (the smallest normal subgroup of G with odd index) has a normal Sylow p-subgroup. In this note, we give a real-version of Thompson's theorem for odd primes by characterising groups with certain divisibility conditions on the degrees of their real-valued irreducible characters in terms of their Sylow structure. As has happened in all the previous cases (except for the extension of Thompson's theorem for real characters and $p = 2$), our proof needs the classification of finite simple groups.

The following is the main result of this work. Notice that again the (necessary) hypothesis on $\mathbf{O}^{2'}(G)$ shows up.

Theorem A. *Suppose that G is a finite group and assume that $\mathbf{O}^{2'}(G) = G$. Let p be an odd prime and suppose that p divides the order of G. Then there is a non-linear real-valued irreducible character $\chi \in \mathrm{Irr}(G)$ with $\chi(1)$ coprime to p.*

2. Theorem A

We will make use of the following result of [7]. We remark that the classification of finite simple groups is involved in it.

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Theorem 2.1. *Let G be a finite group and let p be a prime and assume that the degree of any rational character $\chi \in \mathrm{Irr}(G)$ is one or divisible by p. If one of the following holds*

(1) $p \neq 3$, or

(2) $p = 3$ and G has not a composition factor isomorphic to $S = PSp_{2n}(q)$ with $n \geq 1$, $q = p^f$ with f odd, and $f \neq 1$ if $n = 1$,

then G is solvable.

Proof. If $p \neq 3$ this is Theorem 6.3 of [7]. Suppose now that $p = 3$ and G does not have a composition factor isomorphic to $S = PSp_{2n}(q)$ with $n \geq 1$, $q = p^f$ with f odd, and $f \neq 1$ if $n = 1$. Let M be a minimal normal subgroup of G. By induction, G/M is solvable. Suppose that $M \cong S^n$ with S non-abelian simple group. In Proposition 3.1 of [7] it is proved that if some $\mathrm{Aut}(S)$-orbit $\mathcal{X} \subseteq \mathrm{Irr}(S)$ of size coprime to p satisfies that every element $\alpha \in \mathcal{X}$ is non-linear of degree not divisible by p and extends to a rational character of its stabilizer in $\mathrm{Aut}(S)$, then G has a non-linear rational-valued irreducible character of degree coprime to p (in fact the result in Proposition 3.1 of [7] is more general, but we just need to state it like this). If S is a sporadic simple group, an alternating group or the Tits group, then Lemma 4.1 of [7] shows that there exists an $\mathrm{Aut}(S)$-orbit $\mathcal{X} \subseteq \mathrm{Irr}(S)$ satisfying this. A titanic effort in Section 5 of [7] shows that if S is a group of Lie type, $S \neq PSp_{2n}(q)$ with $n \geq 1$, $q = p^f$ with f odd, and $f \neq 1$ if $n = 1$, then there also exists an $\mathrm{Aut}(S)$-orbit $\mathcal{X} \subseteq \mathrm{Irr}(S)$ verifying this property. Hence G has a non-linear rational-valued irreducible character of degree coprime to p, a contradiction with our hypothesis. Therefore we have that M is elementary abelian, and hence G is solvable, as wanted. □

We will also need the following result, which is obtained by using the proof of Proposition 4.5 of [4]. In the first part of the proof we introduce some notation on simple groups for the reader's convenience.

Lemma 2.2. *Let $S = PSp_{2n}(q)$ with $n \geq 1$, $q = p^f$, $p = 3$, f odd, and $f > 1$ if $n = 1$. Let H be an almost simple group with socle S and $|H/S|$ not divisible by p. Then H has a real-valued irreducible character of degree not divisible by p and not containing S in its kernel.*

Proof. Let $G = PCSp_{2n}(q)$. Then $S = [G, G]$ and there exists a simple algebraic group $\mathcal{G}$ of adjoint type defined over a field of characteristic p and a Frobenius morphism $F : \mathcal{G} \to \mathcal{G}$ such that $G = \mathcal{G}^F$. By the Deligne–Lusztig theory we know that irreducible characters of G are partitioned into Lusztig series that are labeled by conjugacy classes of semisimple elements s in the dual group L, where the pair $(\mathcal{L}, F^*)$ is dual to $(\mathcal{G}, F)$ and $L = \mathcal{L}^{F^*}$. Furthermore if s is any semisimple element of L, then $\mathbf{C}_{\mathcal{L}}(s)$ is connected (because $\mathcal{L}$ is simply connected) and hence the L-conjugacy class s^L corresponds to a unique irreducible character $\chi_s \in \mathrm{Irr}(G)$ of degree coprime to p (in fact $\chi_s(1) = [L : \mathbf{C}_L(s)]_{p'}$). In this case we have $L = \mathrm{SL}_2(q)$ if $n = 1$ and $L = \mathrm{Spin}_{2n+1}(q)$ otherwise. Notice that in both cases $\mathbf{Z}(L)$ is a 2-group. Furthermore we have $\mathrm{Aut}(S) = G : A(S)$, where $A(S)$ is an abelian group (of field automorphisms).

In the first part of the proof of Proposition 4.5 of [4] they choose a semisimple real element $s \in L$ of odd order r, where r is selected conveniently. Let $\chi = \chi_s$ be the corresponding irreducible character of G. Then $\chi(1)$ is not divisible by $p = 3$ and by Proposition 4.3 of [4], we know that χ is real-valued and it restricts irreducibly to $\theta = \chi_S \in \mathrm{Irr}(S)$. Furthermore θ is non-trivial. The last part of the proof of Proposition 4.5 of [4] shows that any automorphism $\sigma \in \mathrm{Aut}(S)$ that fixes θ must belong to G and hence G is the stabilizer of θ in $\mathrm{Aut}(S)$.

Now $\psi = \chi_{H\cap G} \in \mathrm{Irr}(H\cap G)$ is real-valued and lies over θ (hence does not contain S in its kernel). By the Clifford correspondence, ψ^H is a real-valued irreducible character of degree not divisible by 3 and does not contain S in its kernel, as desired.

□

Next we prove Theorem A.

Proof of Theorem A. Suppose that there is an odd prime p dividing $|G|$ and dividing the degrees of all the non-linear real-valued irreducible characters of G. By Theorem 2.1, if G is simple, then either $G = C_r$ for some prime r or $G = PSp_{2n}(q)$ with $n \geq 1$, $q = 3^f$ with f odd, and $f \neq 1$ if $n = 1$. If $G = C_r$, the hypothesis on $\mathbf{O}^{2'}(G)$ forces $r = 2 \neq p$, and we are done. On the other hand, if $G = PSp_{2n}(q)$ then we get a contradiction by Lemma 2.2. Hence we may assume that G is not simple. We proceed by induction on the order of G.

Let M be a minimal normal subgroup of G. Since $\mathbf{O}^{2'}(G/M) = G/M$, if G/M has order divisible by p, then by induction we obtain a non-linear real-valued irreducible character of G/M of degree coprime to p. Since $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(G)$, we get a contradiction, as wanted. Hence G/M is a p'-group. Notice that, in particular, we may assume that M is the unique minimal normal subgroup of G and also that p divides $|M|$.

Suppose first that M is p-elementary abelian and let Q be a Sylow 2-subgroup of G. If $[Q, M] = 1$, then $G/\mathbf{C}_G(M)$ has odd order, and since $\mathbf{O}^{2'}(G) = G$ we have that M is central. By the Schur-Zassenhaus theorem, we have that $G = M \times K$ for some $K \lhd G$. But now $Q \leq K$ and hence G/K is odd. Thus we deduce that $M = 1$, a contradiction.

Thus we may assume that $[Q, M] \neq 1$. In particular, the induced action of Q on $\mathrm{Irr}(M)$ is non-trivial and by Lemma 3.1 of [3], we conclude that there exists a non-trivial character $\lambda \in \mathrm{Irr}(M)$ such that $\lambda^x = \lambda^{-1} = \overline{\lambda}$ for some x in Q. Let ψ be the canonical extension of λ to its stabilizer in G, $I_G(\lambda)$ (see Corollary 8.16 of [2], for instance). Since $I_G(\lambda)$ is the stabilizer of $\overline{\lambda} = \lambda^x$, we have that x normalizes G_λ and ψ^x is the canonical extension of λ^x to $I_G(\lambda)$. But $\overline{\psi}$ is the canonical extension of $\overline{\lambda} = \lambda^x$, and it follows that $\psi^x = \overline{\psi}$. Therefore $\chi = \psi^G \in \mathrm{Irr}(G)$ has degree $|G : I_G(\lambda)|$ not divisible by p and satisfies $\chi = \chi^x = (\psi^x)^G = (\overline{\psi})^G = \overline{\chi}$. By hypothesis, we conclude that χ is linear, so $G = I_G(\lambda)$ and hence $\lambda = \lambda^x = \overline{\lambda} = \lambda^{-1}$, then λ is trivial, a contradiction.

Therefore, we assume that $M = S_1 \times S_2 \times \cdots \times S_n$ is a direct product of n copies of a non-abelian simple group $S = S_1$. By Theorem 2.1 we may assume that $p = 3$ and $S \simeq PSp_{2n}(q)$ with $n \geq 1$, $q = p^f$ with f odd and $f > 1$ if $n = 1$. Let $N = \mathbf{N}_G(S)$, $C = \mathbf{C}_G(S)$ and $L = N/C$. We observe that $S \cong MC/C \trianglelefteq L$. Since $|L : S| = |N : MC|$ is not divisible by 3, by Lemma 2.2 we have that there exists a real-valued character $\theta \in \mathrm{Irr}(L)$ of degree not divisible by 3 which does not contain S in its kernel. Let α be an irreducible constituent of θ_S. Since $S_i \subseteq C \subseteq \ker(\theta)$ for all $1 < i \leq n$, we have that $\mu = \alpha \times 1_{S_2} \times \cdots \times 1_{S_n} \in \mathrm{Irr}(M)$ is an irreducible constituent of θ_M. It follows that $I_G(\mu) \subseteq N$ and since θ lies over μ, by Clifford's correspondence (see Theorem 6.11 of [2], for instance), $\chi = \theta^G \in \mathrm{Irr}(G)$ is a real-valued character of degree $\chi(1) = |G : N|\theta(1)$, which is not divisible by 3, a contradiction.

□

A natural question now is what happens with the conclusion of Theorem A if one replaces real-valued irreducible characters by irreducible characters with values in a given field F. We notice that it remains true if $F = \mathbb{Q}(\xi)$, where ξ is a primitive p-th root of unity. This follows directly from Theorem C of [7] and the hypothesis

$\mathbf{O}^{2'}(G) = G$. However, it does no longer hold if $F = \mathbb{Q}$. For example, if $p = 3$, $G = \mathrm{PSL}_2(27)$ has just one non-trivial rational irreducible characters, of degree 27, and 3 divides $|G|$; if $p > 3$ and $G = D_{2p}$, the dihedral group of $2p$ elements, then $\mathbf{O}^{2'}(G) = G$, G has just two rational irreducible characters, which are linear and p divides $|G|$.

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