

p -BLOCKS RELATIVE TO A CHARACTER OF A NORMAL SUBGROUP

NOELIA RIZO

ABSTRACT. Let G be a finite group, let $N \triangleleft G$, and let $\theta \in \text{Irr}(N)$ be a G -invariant character. We fix a prime p , and we introduce a canonical partition of $\text{Irr}(G|\theta)$ relative to p . We call each member B_θ of this partition a θ -block, and to each θ -block B_θ we naturally associate a conjugacy class of p -subgroups of G/N , which we call the θ -defect groups of B_θ . If N is trivial, then the θ -blocks are the Brauer p -blocks. Using θ -blocks, we can unify the Gluck-Wolf-Navarro-Tiep theorem and Brauer's Height Zero conjecture in a single statement, which, after work of B. Sambale, turns out to be equivalent to the Height Zero conjecture. We also prove that the $k(B)$ -conjecture is true if and only if every θ -block B_θ has size less than or equal the size of any of its θ -defect groups, hence bringing normal subgroups to the $k(B)$ -conjecture.

1. INTRODUCTION

Many of the global-local conjectures in the representation theory of finite groups have been proved or reduced to finite simple groups because, somewhat miraculously, they admit a far more general *projective* version. This projective version always involves an irreducible complex character θ over a normal subgroup N and a statement which only takes into account the irreducible characters of the group lying over θ . When N is the trivial group, one recovers the original conjecture.

Our main concern in this paper is to refine Brauer classical p -blocks, where p is a prime, with respect to a fixed character of a normal subgroup. If N is a normal subgroup of a finite group G , $\theta \in \text{Irr}(N)$ is G -invariant, and we wish to study the block theory of G over θ , sometimes Brauer's p -blocks do not fully perceive some aspects of the character theory of G over θ . Consider, for instance, the Gluck-Wolf-Navarro-Tiep theorem ([GW], [NT]), which is a crucial step in the reduction of Brauer's Height Zero Conjecture to a problem on simple groups ([NS]), and the fact that only one direction of the theorem holds: if p does not divide $\chi(1)/\theta(1)$ for all

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$\chi \in \text{Irr}(G)$ lying over θ , then G/N has abelian Sylow p -subgroups. The converse is not true.

In this paper we fix a prime p , and we introduce a canonical partition of the irreducible characters $\text{Irr}(G|\theta)$ of G that lie over θ (relative to p). We call each member B_θ of this partition a **θ -block** of G , and to each θ -block B_θ we naturally associate a conjugacy class of p -subgroups D_θ/N of G/N , which we call the **θ -defect groups** of B_θ . Both the θ -blocks and their θ -defect groups are defined in terms of some convenient central extensions of G and some projective representations of G associated with θ , and a non-trivial part of this work is to show that they are independent of any choice that has been made in order to define them. As we shall show, each θ -block B_θ is contained in a unique Brauer p -block B of G , and the θ -defect group D_θ/N is contained in DN/N , for some defect group D of B .

Using θ -blocks, we can propose, for instance, the following statement that simultaneously generalizes the Height Zero Conjecture and the Gluck-Wolf-Navarro-Tiep theorem. Recall that n_p is the largest power of p that divides the integer n .

Conjecture A. *Let G be a finite group, let $N \triangleleft G$ and let $\theta \in \text{Irr}(N)$ be G -invariant. Suppose that $B_\theta \subseteq \text{Irr}(G|\theta)$ is a θ -block with θ -defect group D_θ/N . Assume that θ extends to D_θ . Then $(\chi(1)/\theta(1))_p = |G : D_\theta|_p$ for all $\chi \in B_\theta$ if and only if D_θ/N is abelian.*

In the important case where N is central, we shall prove that a θ -block is simply $\text{Irr}(B|\theta)$, where B is a Brauer p -block of G and $\text{Irr}(B|\theta)$ is the set of the complex irreducible characters in B that lie over θ , and that a θ -defect group is DN/N , where D is a defect group of B . Using this, Conjecture A is then equivalent to a projective version of the Height Zero conjecture which seems not to have been noticed before ([MN]). Using the theory of fusion systems, B. Sambale has shown that in fact this projective version of the Height Zero Conjecture is equivalent to Brauer's original one ([S]). Using his work, we can prove the following.

Theorem B. *Conjecture A and Brauer's Height Zero conjecture are equivalent.*

Our second motivation to introduce θ -blocks is to have a better understanding of the celebrated Brauer's $k(B)$ -conjecture. As is well-known, this deep conjecture, that asserts that the number of ordinary characters in a block is less than or equal the size of its defect groups, remains not only unsolved but also unreduced to simple groups. While the relative version of the McKay conjecture was easy to formulate, it does not seem easy how to do the same for the $k(B)$ -conjecture. As we can see, using θ -blocks, this can be done. We believe that statements relating normal subgroups and the $k(B)$ -conjecture might help to discern on this problem.

Theorem C. *The $k(B)$ -conjecture is true for every finite group if and only if whenever $N \triangleleft G$ and $\theta \in \text{Irr}(N)$ is G -invariant, then each θ -block B_θ of G has size less than or equal the size of any of its θ -defect groups.*

As we said, our definition of θ -blocks is related to projective representations, and therefore with blocks of twisted group algebras. Of course, these have been studied before by many (including S. B. Conlon [C], W. Reynolds [R], J. Humphreys [Hu], E. C. Dade [D], or A. Laradji [L] in the p -solvable case). However, our character theoretical approach is new and is specifically tailored to be used in the recent developments of the global-local counting conjectures.

With our approach, we are also able to canonically define θ -Brauer characters of G , which will be class functions defined on the elements of $g \in G$ such that gN is p -regular. Using these, we will prove elsewhere that there is a direct relationship between our θ -blocks and the Külshammer-Robinson N -projective characters defined in [KR]. (See also [Z].) In order to do so, a certain new understanding of Brauer p -blocks is required. Specifically, as in the *Projective Height Zero Conjecture*, we shall need to prove certain *projective* results like the following one, which we believe have independent interest.

Theorem D. *Suppose that Z is a central subgroup of G , and let $\theta \in \text{Irr}(Z)$. Let B be a Brauer p -block of G . Then the decomposition matrix $D_\theta = (d_{\chi\varphi})$, where $\chi \in \text{Irr}(B|\theta)$ and $\varphi \in \text{IBr}(B)$ is not of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Easy examples show that in Theorem D, we cannot replace Z central by a G -invariant character of an abelian $Z \triangleleft G$.

This paper is structured as follows. In Section 2, we review some facts on ordinary blocks. In Section 3, we give the necessary background on projective representations and character triples. In Section 4 we give the definitions of θ -blocks and θ -defect groups. In Section 5 we present some properties of the θ -blocks. In particular, we prove a θ -version of a classical theorem on blocks: if $\chi \in \text{Irr}(B_\theta)$, then $\chi(g) = 0$ if g_p is not G -conjugate to any element of D_θ . In Section 6, we prove Theorem D. In Section 7, we prove Theorems B and C.

Further applications of θ -blocks will appear elsewhere.

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2. PRELIMINARIES ON p -BLOCKS

We follow the notation of [Is] for characters, and the notation of [N] for blocks. We denote by \mathbf{R} the ring of algebraic integers in \mathbb{C} , and we choose a maximal ideal M of \mathbf{R} containing $p\mathbf{R}$. Let $F = \mathbf{R}/M$, an algebraically closed field of characteristic p , and let $*$: $\mathbf{R} \rightarrow F$ be the canonical ring homomorphism.

In this paper, sometimes we identify a p -block B of a finite group G with the set $\text{Irr}(B)$ of its irreducible complex characters. Recall that two irreducible characters $\chi, \psi \in \text{Irr}(G)$ of a finite group G are in the same block B of G if and only if $\omega_\chi(\hat{K})^* = \omega_\psi(\hat{K})^*$ for every conjugacy class K of G , where

$$\omega_\chi(\hat{K}) = \frac{|K|\chi(x_K)}{\chi(1)},$$

and x_K is any fixed element of K . We denote by λ_B the corresponding algebra homomorphism $\mathbf{Z}(FG) \rightarrow F$. Thus $\lambda_B(\hat{K}) = \omega_\chi(\hat{K})^*$ if $\chi \in B$. This homomorphism is also denoted by λ_χ . Also $\delta(B)$ is the set of defect groups of the block B .

We write $G^{p'}$ for the set of p -regular elements in G , and $\text{IBr}(G)$ is the set of irreducible Brauer characters of G (calculated with respect to our fixed maximal ideal M). If B is a p -block, we write $\text{IBr}(B)$ to denote the set of irreducible Brauer characters lying in B .

We shall need some basic facts on Brauer p -blocks which we prove in this section.

Lemma 2.1. *Let B be a p -block of G and let μ be a linear character of G . Then*

$$\text{Irr}(\mu B) = \{\mu\chi \mid \chi \in \text{Irr}(B)\}$$

is a p -block. Also, B and μB have the same defect groups.

Proof. Since $\mu(x)$ is a root of unity for $x \in G$, then $\mu(x)^*\mu(x^{-1})^* = 1$. Hence, it is clear that if $\chi, \psi \in \text{Irr}(G)$, then $\lambda_\chi = \lambda_\psi$ if and only if $\lambda_{\mu\chi} = \lambda_{\mu\psi}$, and the first part follows. Now, let K be a defect class of B (see page 82 of [N]). Then $\lambda_B(\hat{K}) \neq 0$ and $a_B(\hat{K}) \neq 0$. Notice that $\lambda_{\mu B}(\hat{K}) = \mu(x_K)^*\lambda_B(\hat{K})$, and $a_{\mu B}(\hat{K}) = \mu(x_K^{-1})^*a_B(\hat{K})$, where $x_K \in K$. Since $\mu(x_K)^* \neq 0$, the result follows from Corollary 4.5 of [N]. \square

If N is normal in G , following [Is] and [N] we view the characters of G/N as characters of G containing N in their kernel. By the remarks preceding Theorem 7.6 of [N], recall that every block of G/N is contained in a block of G . If $\chi \in \text{Irr}(G)$, then we denote by $\text{bl}(\chi)$ the block of G containing χ .

Lemma 2.2. *Let $Z \leq G$, with $Z = Z_p \times K$, where K is a normal p' -subgroup of G and Z_p is a central p -subgroup of G . Let $\alpha \in \text{Irr}(G)$ with $Z \subseteq \ker(\alpha)$ and write $\bar{\alpha}$ for the character α viewed as character of G/Z .*

- (a) *Let $\beta \in \text{Irr}(G)$ with $Z \subseteq \ker(\beta)$. Then $\text{bl}(\alpha) = \text{bl}(\beta)$ if and only if $\text{bl}(\bar{\alpha}) = \text{bl}(\bar{\beta})$.*
- (b) *We have that*

$$\delta(\text{bl}(\bar{\alpha})) = \{PZ/Z \mid P \in \delta(\text{bl}(\alpha))\} = \{P/Z_p \mid P \in \delta(\text{bl}(\alpha))\}.$$

Proof. (a) It is clear that if $\text{bl}(\bar{\alpha}) = \text{bl}(\bar{\beta})$, then $\text{bl}(\alpha) = \text{bl}(\beta)$. We need to prove the converse. We proceed by induction on $|G|$. By Theorem 9.9(c) of [N] we may assume that Z is a central p -group. The result follows by Theorem 7.6 of [N].

For (b), we proceed by induction on $|G|$. Let $\hat{\alpha}$ be the character α viewed as a character of G/K . By Theorem 9.9(c) of [N], we have that

$$\delta(\text{bl}(\hat{\alpha})) = \{PK/K \mid P \in \delta(\text{bl}(\alpha))\}.$$

If $K > 1$, since $G/Z \cong \frac{G/K}{Z/K}$, using induction we are done. Hence, we may assume that $K = 1$. In this case, Z is a central p -group. The result now follows by Theorem 9.10 of [N].

□

Suppose $\alpha : \hat{G} \rightarrow G$ is a surjective group homomorphism with kernel Z . If $\psi \in \text{Irr}(G)$, whenever is convenient, we denote by ψ^α the unique irreducible character of \hat{G} such that $\psi^\alpha(x) = \psi(\alpha(x))$ for $x \in \hat{G}$. Notice that $Z \subseteq \ker(\psi^\alpha)$.

Corollary 2.3. *Suppose that $\alpha : \hat{G} \rightarrow G$ is an onto homomorphism with $\ker(\alpha) = Z \subseteq \mathbf{Z}(\hat{G})$.*

- (a) *If $\chi_i \in \text{Irr}(G)$, then χ_1, χ_2 lie in the same block of G if and only if χ_1^α and χ_2^α lie in the same block of \hat{G} .*
- (b) *Suppose that $L \leq G$ and let $\gamma \in \text{Irr}(L)$. If $\hat{L} = \alpha^{-1}(L)$, let $\hat{\gamma} = \gamma^{\alpha|_{\hat{L}}} \in \text{Irr}(\hat{L})$. Then $[(\chi^\alpha)_{\hat{L}}, \hat{\gamma}] = [\chi_L, \gamma]$ for $\chi \in \text{Irr}(G)$.*
- (c) *Suppose that $\chi \in \text{Irr}(G)$, let B be the block of χ , and let \hat{B} be the block of χ^α . If \hat{D} is a defect group of \hat{B} , then $\alpha(\hat{D})$ is a defect group of B .*

Proof. Part (a) is a direct consequence of Lemma 2.2(a). Part (b) is straightforward. To prove part (c), define $\bar{\alpha} : \hat{G}/Z \rightarrow G$ to be the associated isomorphism. Since $Z \subseteq \ker(\chi^\alpha)$, by Lemma 2.2(b), we have that $\hat{D}Z/Z$ is a defect group of the block of χ^α viewed as a character of \hat{G}/Z . Since $\alpha(\hat{D}) = \bar{\alpha}(\hat{D}Z/Z)$, the result follows. □

We shall use Gallagher's Corollary 6.18 of [Is] in several parts of this paper. Recall that if $N \triangleleft G$, $\chi \in \text{Irr}(G)$ and $\chi_N = \theta$, then $\beta \mapsto \beta\chi$ defines a bijection $\text{Irr}(G/N) \rightarrow$

$\text{Irr}(G|\theta)$, where $\text{Irr}(G|\theta)$ denotes the set of irreducible constituents of the induced character θ^G .

Lemma 2.4. *Let $N \triangleleft G$ and let $\chi \in \text{Irr}(G)$. Suppose that $\chi_N = \theta \in \text{Irr}(N)$. Let $\chi_1, \chi_2 \in \text{Irr}(G|\theta)$ and write $\chi_i = \beta_i \chi$, for $i = 1, 2$, where $\beta_i \in \text{Irr}(G/N)$. Suppose that β_1 and β_2 lie in the same p -block of G/N . Then χ_1 and χ_2 lie in the same p -block of G . Also, if $\beta \in \text{Irr}(G/N)$ and P/N is a defect group of $\text{bl}(\beta)$, then $P \subseteq DN$, for some defect group D of $\text{bl}(\beta\chi)$.*

Proof. Let K be a conjugacy class of G and let $x \in K$. Write $H/N = \mathbf{C}_{G/N}(xN)$, let L be the conjugacy class of H containing x , and let S be the conjugacy class of G/N containing xN . Then, by Lemma 2.2 of [NS], we have that

$$\lambda_{\chi_1}(\hat{K}) = \lambda_{\chi\beta_1}(\hat{K}) = \lambda_{\chi_H}(\hat{L})\lambda_{\beta_1}(\hat{S}).$$

Since β_1 and β_2 lie in the same p -block of G/N , we have that

$$\lambda_{\beta_1}(\hat{S}) = \lambda_{\beta_2}(\hat{S}),$$

and hence, again by Lemma 2.2 of [NS] we have that

$$\lambda_{\chi_1}(\hat{K}) = \lambda_{\chi_H}(\hat{L})\lambda_{\beta_1}(\hat{S}) = \lambda_{\chi_H}(\hat{L})\lambda_{\beta_2}(\hat{S}) = \lambda_{\chi\beta_2}(\hat{K}) = \lambda_{\chi_2}(\hat{K}).$$

Hence χ_1 and χ_2 lie in the same p -block of G . The second part is Proposition 2.5(b) of [NS]. \square

3. REVIEWING PROJECTIVE REPRESENTATIONS

To define the θ -blocks we need some background on projective representations. We follow Chapter 11 of [Is] and Chapter 5 of [N2]. Recall that a complex **projective representation** of a finite group G is a map

$$\mathcal{P} : G \rightarrow \text{GL}_n(\mathbb{C})$$

such that for every $x, y \in G$ there is some $\alpha(x, y) \in \mathbb{C}^\times$ satisfying

$$\mathcal{P}(x)\mathcal{P}(y) = \alpha(x, y)\mathcal{P}(xy).$$

The function $\alpha : G \times G \rightarrow \mathbb{C}^\times$ is called the **factor set** of \mathcal{P} .

If G is a finite group, $N \triangleleft G$, and $\theta \in \text{Irr}(N)$ is G -invariant, then we say that (G, N, θ) is a **character triple**. The theory of character triples and their isomorphisms was developed by I. M. Isaacs, and we refer to Chapter 11 of [Is] for their properties.

If (G, N, θ) is a character triple, we say that a projective representation of G is **associated** with θ if

- (a) \mathcal{P}_N is an ordinary representation of N affording θ , and
- (b) $\mathcal{P}(ng) = \mathcal{P}(n)\mathcal{P}(g)$ and $\mathcal{P}(gn) = \mathcal{P}(g)\mathcal{P}(n)$ for $g \in G$ and $n \in N$.

We will need the following later.

Lemma 3.1. *Suppose that (G, N, θ) is a character triple, and let \mathcal{P} be a projective representation of G associated with θ with factor set α . Then*

- (a) $\alpha(1, 1) = \alpha(g, n) = \alpha(n, g) = 1$ for $n \in N, g \in G$.
- (b) $\alpha(xn, ym) = \alpha(x, y)$ for $x, y \in G, n, m \in N$.

Proof. This is Lemma 11.5 and Theorem 11.7 of [Is]. See also Lemma 5.3 of [N2]. \square

An important fact about projective representations is that given a character triple (G, N, θ) , there always exists a projective representation associated with θ such that its factor set has roots of unity values.

Theorem 3.2. *Let (G, N, θ) be a character triple. Then there exists a projective representation associated with θ with factor set α such that*

$$\alpha(x, y)^{|G|_{\theta(1)}} = 1$$

for all $x, y \in G$.

Proof. See for instance Theorem 8.2 of [I] or Theorem 5.5 of [N2]. \square

Using such a projective representation \mathcal{P} , it is possible to associate to each character triple (G, N, θ) a new finite group \hat{G} , a finite central extension of G which only depends on \mathcal{P} . This finite group \hat{G} contains N as a normal subgroup, and an irreducible character $\tau \in \text{Irr}(\hat{G})$ that extends θ . The next theorem explains exactly how to do this.

Theorem 3.3. *Let (G, N, θ) be a character triple and let \mathcal{P} be a projective representation of G associated with θ such that the factor set α of \mathcal{P} only takes roots of unity values. Let $Z \leq \mathbb{C}^\times$ be the subgroup generated by the values of α . Let $\hat{G} = \{(g, z) \mid g \in G, z \in Z\}$ with the multiplication given as follows:*

$$(x, a)(y, b) = (xy, \alpha(x, y)ab).$$

Then \hat{G} is a finite group. Besides, if we identify N with $N \times 1$ and Z with $1 \times Z$, we have that the following hold.

- (a) $N \triangleleft \hat{G}$, $Z \subseteq \mathbf{Z}(\hat{G})$, and $\hat{N} = N \times Z \triangleleft \hat{G}$. Moreover, if $\pi : \hat{G} \rightarrow G$ is given by $(g, z) \mapsto g$, then π is an onto homomorphism with kernel Z . Also, if $N \subseteq \mathbf{Z}(G)$, then $\hat{N} \subseteq \mathbf{Z}(\hat{G})$.
- (b) The function $\hat{\mathcal{P}}(g, z) = z\mathcal{P}(g)$ defines an irreducible representation of \hat{G} whose character $\tau \in \text{Irr}(\hat{G})$ extends θ . In fact, $\tau(n, z) = z\theta(n)$ for $n \in N$ and $z \in Z$. In particular, if $\hat{\theta} = \theta \times 1_Z \in \text{Irr}(\hat{N})$, and $\hat{\lambda} \in \text{Irr}(\hat{N})$ is defined by $\hat{\lambda}(n, z) = z^{-1}$, then $\hat{\lambda}$ is a linear \hat{G} -invariant character with $N = \ker(\hat{\lambda})$ and $\hat{\lambda}^{-1}\hat{\theta}$ extends to $\tau \in \text{Irr}(\hat{G})$.

Proof. See Theorem 11.28 of [Is] or Theorem 5.6 of [N2]. The properties of the factor set α that we have listed in Lemma 3.1 are essential to prove (a). \square

Given a character triple (G, N, θ) , we call the group \hat{G} defined in Theorem 3.3 a **representation group for (G, N, θ) associated with \mathcal{P}** . We also say that $\tau \in \text{Irr}(\hat{G})$ is the character of \hat{G} **associated with \mathcal{P}** . By Theorem 3.3(b), we have that $\tau_N = \theta$.

In order to define the θ -blocks the following is essential. We assume the reader is familiar with the notion of character triple isomorphism (see Definition 11.23 of [Is]).

Theorem 3.4. *Let (G, N, θ) be a character triple and let \mathcal{P} be a projective representation of G associated with θ . Let \hat{G} be a representation group for (G, N, θ) associated with \mathcal{P} . Then (G, N, θ) and $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ are isomorphic character triples.*

Proof. See Theorem 11.28 of [Is] or Corollary 5.9 of [N2]. \square

We shall frequently use how this character triple isomorphism is constructed. Let $\chi \in \text{Irr}(G|\theta)$. We show how to construct $\chi^* \in \text{Irr}(\hat{G}/N|\hat{\lambda})$. Let $\pi : \hat{G} \rightarrow G$ be the homomorphism $(g, z) \mapsto g$, which has kernel Z . Since π induces an isomorphism $\hat{G}/Z \rightarrow G$, there is a unique $\chi^\pi \in \text{Irr}(\hat{G})$ such that $\chi^\pi(g, z) = \chi(g)$ for all $g \in G, z \in Z$. Since χ lies over θ notice that χ^π lies over $\hat{\theta} = \theta \times 1_Z$, and in particular over θ . Now by Theorem 3.3(b), the character τ extends θ . By Gallagher's Corollary 6.17 of [Is], there exists a unique $\chi^* \in \text{Irr}(\hat{G}/N)$ such that $\chi^\pi = \chi^* \tau$. (Recall that we view the characters of H/N as characters of H that contain N in its kernel.) Now, evaluating in $(1, z)$ for $z \in Z$, we easily check that $\chi^* \in \text{Irr}(\hat{G}/N|\hat{\lambda})$. The fact that $\chi \mapsto \chi^*$ defines an isomorphism of character triples is the content of the proof of Theorem 3.4. (The same construction can be done for every subgroup $N \leq U \leq G$ in place of G .) Also, recall that $\hat{\lambda}(n, z) = z^{-1}$ for $n \in N$ and $z \in Z$.

We say that the character triple $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ is a **standard isomorphic character triple** for (G, N, θ) given by \mathcal{P} , and that the bijective map $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(\hat{G}/N|\hat{\lambda})$ that we have constructed, is the **standard bijection**.

4. θ -BLOCKS

If (G, N, θ) is a character triple, we are now ready to define the θ -blocks of G , and their θ -defect groups.

Definition 4.1. *Let (G, N, θ) be a character triple. Let \hat{G} be a representation group for (G, N, θ) and let $\pi : \hat{G} \rightarrow G$ be the canonical homomorphism $(g, z) \mapsto g$ with kernel Z . Let $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(\hat{G}/N|\hat{\lambda})$ be the associated standard bijection. We say*

that a non-empty subset $B_\theta \subseteq \text{Irr}(G|\theta)$ is a θ -**block** of G if there exists a p -block \hat{B} of \hat{G}/N such that

$$B_\theta^* = \{\chi^* \mid \chi \in B_\theta\} = \text{Irr}(\hat{B}|\hat{\lambda}).$$

If \hat{D}/N is a defect group of \hat{B} , then we say that $\pi(\hat{D})/N$ is a θ -**defect group** of B_θ .

Of course, note that the definition of θ -blocks depends on the choice of the standard isomorphic character triple and therefore on the choice of the projective representation associated with θ . The same happens with the θ -defect groups. Our main result in this section is that θ -blocks are in fact canonically defined, and that all the θ -defect groups are G/N -conjugate. The following result is the key to proving that.

Theorem 4.2. *Let (G, N, θ) be a character triple. Let $\mathcal{P}_1, \mathcal{P}_2$ be projective representations of G associated with θ , with factor sets α_1 and α_2 , respectively, whose values are roots of unity. Let \hat{G}_i be the representation group associated with \mathcal{P}_i . Let $(\hat{G}_1/N, \hat{N}_1/N, \hat{\lambda}_1)$ and $(\hat{G}_2/N, \hat{N}_2/N, \hat{\lambda}_2)$ be the standard isomorphic character triples given by \mathcal{P}_1 and \mathcal{P}_2 , respectively. Let $\hat{G} = G \times Z_1 \times Z_2$ and define the product*

$$(g, z_1, z_2)(h, z'_1, z'_2) = (gh, \alpha_1(g, h)z_1z'_1, \alpha_2(g, h)z_2z'_2).$$

Then the following hold.

- (a) \hat{G} is a finite group, $N \times 1 \times 1$ is a normal subgroup of \hat{G} (which we identify with N), and $1 \times Z_1 \times Z_2$ is a central subgroup of \hat{G} (which we identify with $Z_1 \times Z_2$). Also, $\hat{N} = N \times Z_1 \times Z_2$ is a normal subgroup of \hat{G} and \hat{N}/N is central in \hat{G}/N .
- (b) The maps $\rho_1 : \hat{G} \rightarrow \hat{G}_1$ and $\rho_2 : \hat{G} \rightarrow \hat{G}_2$ given by $(g, z_1, z_2) \mapsto (g, z_1)$ and $(g, z_1, z_2) \mapsto (g, z_2)$ are surjective homomorphisms with kernels Z_2 and Z_1 , respectively.
- (c) Suppose that $\tau_i \in \text{Irr}(\hat{G}_i)$ is the character associated with \mathcal{P}_i , and let $\tau_i^{\rho_i} \in \text{Irr}(\hat{G})$ be the corresponding character of \hat{G} . Then there exists a linear character $\beta \in \text{Irr}(\hat{G}/N)$ such that

$$\tau_1^{\rho_1} = \beta\tau_2^{\rho_2}.$$

- (d) Let $\chi \in \text{Irr}(G|\theta)$ and let $\chi_i^* \in \text{Irr}(\hat{G}_i/N|\hat{\lambda}_i)$ be the image of χ under the standard bijection. Let $\hat{\chi}_i = (\chi_i^*)^{\rho_i} \in \text{Irr}(\hat{G}/N)$. Then $\beta\hat{\chi}_1 = \hat{\chi}_2$. As a consequence, if \hat{B}_i is the block of \hat{G}/N containing $\hat{\chi}_i$, then $\hat{B}_2 = \beta\hat{B}_1$.
- (e) Let B_i^* be the block of \hat{G}_i/N containing χ_i^* . Then the map $\psi \mapsto \psi^{\rho_i}$ is a bijection from $\text{Irr}(B_i^*|\hat{\lambda}_i)$ to $\text{Irr}(\hat{B}_i|\tilde{\lambda}_i)$, where $\tilde{\lambda}_1(n, z_1, z_2) = \hat{\lambda}_1(1, z_1) = z_1^{-1}$ and $\tilde{\lambda}_2(n, z_1, z_2) = \hat{\lambda}_2(1, z_2) = z_2^{-1}$ are linear characters of \hat{N}/N .
- (f) The map $\psi \mapsto \beta\psi$ is a bijection from $\text{Irr}(\hat{B}_1|\tilde{\lambda}_1)$ to $\text{Irr}(\hat{B}_2|\tilde{\lambda}_2)$. In particular, $|\text{Irr}(B_1^*|\hat{\lambda}_1)| = |\text{Irr}(B_2^*|\hat{\lambda}_2)|$.

(g) Let $\pi_i : \hat{G}_i \rightarrow G$ be the canonical homomorphism $(g, z_i) \mapsto g$ with kernel Z_i . If \hat{D}_i/N is defect group of B_i^* , then $\pi_1(\hat{D}_1)$ and $\pi_2(\hat{D}_2)$ are G -conjugate.

Proof. Using Lemma 3.1, parts (a) and (b) are straightforward. We prove (c). Since \mathcal{P}_1 and \mathcal{P}_2 are projective representations of G associated to θ , by Theorem 11.2 of [Is] we know that there exists $\xi : G \rightarrow \mathbb{C}^\times$ with $\xi(1) = 1$, constant on the cosets of N , such that $\mathcal{P}_2 = \xi\mathcal{P}_1$, and the factor sets α_1 and α_2 are related in this way

$$\alpha_2(g, h) = \alpha_1(g, h)\xi(g)\xi(h)\xi(gh)^{-1}.$$

Now $\tau_i \in \text{Irr}(\hat{G}_i)$ is the character afforded by the irreducible representation $\hat{\mathcal{P}}_i$, which is defined by $\hat{\mathcal{P}}_i(g, z_i) = z_i\mathcal{P}_i(g)$, for $z_i \in Z_i$ and $g \in G$. Then, using that $\mathcal{P}_2 = \xi\mathcal{P}_1$, we have that

$$\tau_1(g, z_1) = z_1z_2^{-1}\xi(g)^{-1}\tau_2(g, z_2)$$

for $g \in G$ and $z_i \in Z_i$. It is straightforward to prove that the function $\beta : \hat{G} \rightarrow \mathbb{C}^\times$ defined by

$$\beta(g, z_1, z_2) = z_1z_2^{-1}\xi(g)^{-1}$$

is a linear character of \hat{G} that contains N in its kernel.

By definition, we have that $\tau_1^{\rho_1}(g, z_1, z_2) = \tau_1(g, z_1)$ and $\tau_2^{\rho_2}(g, z_1, z_2) = \tau_2(g, z_2)$. Therefore $\tau_1^{\rho_1} = \beta\tau_2^{\rho_2}$, as desired. This proves (c).

Let us denote by $\pi_i : \hat{G}_i \rightarrow G$ the homomorphism $(g, z_i) \mapsto g$. Recall that, by definition, $\chi_i^* \in \text{Irr}(\hat{G}_i/N)$ is the unique character satisfying $\chi^{\pi_i} = \chi_i^*\tau_i$. That is

$$\chi(g) = \chi^{\pi_i}(g, z_i) = \chi_i^*(g, z_i)\tau_i(g, z_i)$$

for $g \in G$ and $z_i \in Z_i$. By definition, $\hat{\chi}_1(g, z_1, z_2) = \chi_1^*(g, z_1)$ and $\hat{\chi}_2(g, z_1, z_2) = \chi_2^*(g, z_2)$. In particular, $\hat{\chi}_i \in \text{Irr}(\hat{G})$ contains N in its kernel. Notice that we have that $\tau_1^{\rho_1}\hat{\chi}_1 = \tau_2^{\rho_2}\hat{\chi}_2$. Hence,

$$\beta\hat{\chi}_1\tau_2^{\rho_2} = \hat{\chi}_2\tau_2^{\rho_2}.$$

Since $\tau_2^{\rho_2}$ extends $\theta \in \text{Irr}(N)$ and $\beta\hat{\chi}_1, \hat{\chi}_2 \in \text{Irr}(\hat{G}/N)$, by Gallagher's Corollary 6.18 of [Is], we have that

$$\beta\hat{\chi}_1 = \hat{\chi}_2.$$

Using Lemma 2.1, part (d) easily follows.

Next we prove part (e). Since $\rho_1(N) = N$, then ρ_1 uniquely defines an onto homomorphism $\tilde{\rho}_1 : \hat{G}/N \rightarrow \hat{G}_1/N$ with kernel $NZ_2/N \subseteq \mathbf{Z}(\hat{G}/N)$. Since $N \subseteq \ker(\chi_1^*)$, then notice that $\hat{\chi}_1 = (\chi_1^*)^{\tilde{\rho}_1}$. Now NZ_1/N is a subgroup of \hat{G}_1/N , and its inverse image under $\tilde{\rho}_1$ is \hat{N}/N . Also, the character corresponding to $\hat{\lambda}_1$ under $\tilde{\rho}_1$ is $\tilde{\lambda}_1$. By Corollary 2.3 (a) and (b), we have that $\psi \mapsto \psi^{\rho_i}$ is a bijection from $\text{Irr}(B_i^*|\hat{\lambda}_i)$

to $\text{Irr}(\hat{B}_i|\tilde{\lambda}_i)$. (Notice that $\psi^{\rho_i} = \psi^{\tilde{\rho}_i}$ because all of our characters have N in their kernel).

Now we prove part (f). By using their definitions (and the fact that $\xi(n) = 1$ for $n \in N$), we check that $\beta_{\tilde{N}}\tilde{\lambda}_1 = \tilde{\lambda}_2$. Therefore, multiplication by the linear character β sends bijectively $\text{Irr}(\hat{B}_1|\tilde{\lambda}_1) \rightarrow \text{Irr}(\hat{B}_2|\tilde{\lambda}_2)$.

Finally, we prove part (g). As in part (e), we have that $\tilde{\rho}_i : \hat{G}/N \rightarrow \hat{G}_i/N$ is an onto homomorphism, with central kernel, such that the map $\psi \mapsto \psi^{\tilde{\rho}_i}$ is a bijection from $\text{Irr}(B_i^*|\lambda_i)$ to $\text{Irr}(\hat{B}_i|\tilde{\lambda}_i)$. Let E_i/N be a defect group of \hat{B}_i . Since $\hat{B}_2 = \beta\hat{B}_1$, we may assume that $E_i = E$ for $i = 1, 2$. by Lemma 2.1. By Corollary 2.3(c), we have that $\tilde{\rho}_i(E/N)$ is a defect group of B_i^* . Hence $\tilde{\rho}_i(E/N) = (\hat{D}_i/N)^{(g_i,1)}$ for some $g_i \in G$ (using that Z_i is central in \hat{G}_i). Now, since $\pi_i(N) = N$, we have that π_i uniquely determines an onto homomorphism $\tilde{\pi}_i : \hat{G}_i/N \rightarrow G/N$. We easily check that $\tilde{\pi}_1 \circ \tilde{\rho}_1 = \tilde{\pi}_2 \circ \tilde{\rho}_2$. Then

$$\pi_1(\hat{D}_1)^{g_1} = \pi_2(\hat{D}_2)^{g_2},$$

as desired. □

We can now prove the main result of this section.

Theorem 4.3. *Suppose that $N \triangleleft G$, and $\theta \in \text{Irr}(N)$ is G -invariant. Then the θ -blocks of G are well defined. Furthermore, the set of θ -defect groups is a G/N -conjugacy class of p -subgroups of G/N .*

Proof. Let (G, N, θ) be a character triple and let \mathcal{P}_1 and \mathcal{P}_2 be projective representations associated with θ . Let $(\hat{G}_1/N, \hat{N}_1/N, \hat{\lambda}_1)$ and $(\hat{G}_2/N, \hat{N}_2/N, \hat{\lambda}_2)$ be the standard isomorphic character triples given by \mathcal{P}_1 and \mathcal{P}_2 respectively. Let $\pi_i : \hat{G}_i \rightarrow G$ be the homomorphism $\pi_i(g, z_i) = g$, and let $\tau_i \in \text{Irr}(\hat{G}_i)$ be the character associated with \mathcal{P}_i . Recall that if $\chi \in \text{Irr}(G|\theta)$, then $\chi^{\pi_i} = \chi_i^* \tau_i$, for some uniquely defined $\chi_i^* \in \text{Irr}(\hat{G}_i/N)$. The map $\chi \mapsto \chi_i^*$ from $\text{Irr}(G|\theta) \rightarrow \text{Irr}(\hat{G}_i/N|\hat{\lambda}_i)$ is the standard bijection.

Let $A_1, A_2 \subseteq \text{Irr}(G|\theta)$ be such that $A_1^* = \{\varphi_1^* \mid \varphi \in A_1\} = \text{Irr}(B_1^*|\hat{\lambda}_1)$ and $A_2^* = \{\varphi_2^* \mid \varphi \in A_2\} = \text{Irr}(B_2^*|\hat{\lambda}_2)$, where B_i^* is a block of \hat{G}_i/N . Suppose that $\chi \in A_1 \cap A_2$. We wish to prove that $A_1 = A_2$.

In order to do so, we construct the group \hat{G} in Theorem 4.2, and consider the group homomorphisms $\rho_i : \hat{G} \rightarrow \hat{G}_i$, in Theorem 4.2(b). By part (c) of this theorem, there is a linear character $\beta \in \text{Irr}(\hat{G}/N)$ satisfying

$$\tau_1^{\rho_1} = \beta \tau_2^{\rho_2}.$$

As in Theorem 4.2(d), let $\hat{\chi}_i = (\chi_i^*)^{\rho_i} \in \text{Irr}(\hat{G}/N)$, and let \hat{B}_i be the block of \hat{G}/N containing $\hat{\chi}_i$. By Theorem 4.2(d), we have that $\hat{B}_2 = \beta\hat{B}_1$. By Theorem 4.2(f), $|A_1^*| = |A_2^*|$, and therefore $|A_1| = |A_2|$. We only need to prove that $A_1 \subseteq A_2$, for instance.

Let $\varphi_1 \in A_1$. Now, $\varphi_1^* \in A_1^* = \text{Irr}(B_1^*|\hat{\lambda}_1)$, and by Theorem 4.2(e) we have that $\hat{\varphi}_1 = (\varphi_1^*)^{\rho_1} \in \text{Irr}(\hat{B}_1|\tilde{\lambda}_1)$. By Theorem 4.2(f), $\beta\hat{\varphi}_1 \in \text{Irr}(\hat{B}_2|\tilde{\lambda}_2)$. By Theorem 4.2(e), let $\varphi_2 \in A_2$ be such that $\beta\hat{\varphi}_1 = (\varphi_2^*)^{\rho_2}$. We claim that $\varphi_1 = \varphi_2$. Recall that $\tau_i\varphi_i^* = \varphi_i^{\pi_i}$ and that $\tau_1^{\rho_1} = \beta\tau_2^{\rho_2}$. If $g \in G$, then we have that

$$\begin{aligned} \varphi_1(g) &= \varphi_1^{\pi_1}(g, 1) = \tau_1(g, 1)\varphi_1^*(g, 1) \\ &= \tau_1^{\rho_1}(g, 1, 1)\hat{\varphi}_1(g, 1, 1) \\ &= \beta(g, 1, 1)\tau_2^{\rho_2}(g, 1, 1)\hat{\varphi}_1(g, 1, 1) \\ &= \tau_2(g, 1)(\beta\hat{\varphi}_1)(g, 1, 1) \\ &= \tau_2(g, 1)(\varphi_2^*)^{\rho_2}(g, 1, 1) \\ &= \tau_2(g, 1)\varphi_2^*(g, 1) \\ &= \varphi_2^{\pi_2}(g, 1) \\ &= \varphi_2(g), \end{aligned}$$

as desired. This completes the proof of the first part of the theorem. The second part easily follows from Theorem 4.2(g). It is elementary to show that the θ -defect groups are p -subgroups of G/N . \square

5. PROPERTIES OF θ -BLOCKS

The following gives us some key properties of θ -blocks.

Theorem 5.1. *Let (G, N, θ) be a character triple. Let B_θ be a θ -block of G , and let D_θ/N be a θ -defect group of B_θ .*

- (a) *There is a p -block B of G such that B_θ is contained in the set $\text{Irr}(B|\theta)$. Also, there is a defect group D of B such that $D_\theta \subseteq DN$.*
- (b) *If $N \subseteq \mathbf{Z}(G)$, then there is a p -block B of G and a defect group D of B such that $B_\theta = \text{Irr}(B|\theta)$, and $D_\theta = DN$.*
- (c) *If θ has an extension $\chi \in \text{Irr}(G)$, then there is a p -block \bar{B} of G/N and a defect group DN/N of \bar{B} such that $B_\theta = \{\gamma\chi \mid \gamma \in \text{Irr}(\bar{B})\}$ and $D_\theta/N = DN/N$.*
- (d) *If G/N is a p -group, then $B_\theta = \text{Irr}(G|\theta)$ and $D_\theta/N = G/N$.*

Proof. Let \hat{G} be a representation group associated with (G, N, θ) , with associated character $\tau \in \text{Irr}(\hat{G})$. Recall that $\tau_N = \theta$. Let $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(\hat{G}/N|\hat{\lambda})$ be the standard bijection. Let $\pi : \hat{G} \rightarrow G$ be the homomorphism $(g, z) \mapsto g$. Since $\pi(N) = N$, let $\hat{\pi} : \hat{G}/N \rightarrow G/N$ be the corresponding onto homomorphism. Notice that \hat{G}/N is a central extension of G/N .

By definition, there is a Brauer p -block \hat{B} of \hat{G}/N such that $(B_\theta)^* = \text{Irr}(\hat{B}|\hat{\lambda})$. Recall that $\chi^\pi = \chi^*\tau$ for $\chi \in \text{Irr}(G|\theta)$.

Now, fix $\chi \in B_\theta$ and let B be the p -block of G containing χ . We claim that $B_\theta \subseteq \text{Irr}(B|\theta)$. Indeed, let $\psi \in B_\theta$. Then $\chi^*, \psi^* \in \hat{B}$. Since $\tau_N = \theta$ and $\chi^\pi = \tau\chi^*$, $\psi^\pi = \tau\psi^*$, by Lemma 2.4 we have that χ^π and ψ^π lie in the same p -block of \hat{G} . By Corollary 2.3, χ, ψ lie in the same p -block of G . This proves the first part of (a). If \hat{D}/N is a defect group of \hat{B} , by Lemma 2.4 we have that $\hat{D} \subseteq EN$ for some defect group E of the block of χ^π . Now, $\pi(E)$ is a defect group of the block of χ by Corollary 2.3(c), and $\pi(\hat{D}) \subseteq \pi(E)N$. This proves the second part of (a). Notice now that if N is central, then τ is linear and the defect groups of the block of $\chi^\pi = \tau\chi^*$ are the defect groups of the block of χ^* (multiplying by τ^{-1} and using Lemma 2.1). Since N is central in \hat{G} by Theorem 3.3(a), we have that $\hat{D} = EN$ by Lemma 2.2(b).

Next, we complete the proof of part (b). Suppose N is central and that $\gamma \in \text{Irr}(B|\theta)$. In particular, τ is linear. Write $\gamma^\pi = \gamma^*\tau$, for some $\gamma^* \in \text{Irr}(\hat{G}/N|\hat{\lambda})$. Now, since γ and χ lie in the same p -block of G , we have that γ^π and χ^π lie in the same p -block of \hat{G} by Corollary 2.3. Therefore $\gamma^*\tau$ and $\chi^*\tau$ lie in the same p -block of \hat{G} . By Lemma 2.1, multiplying by τ^{-1} , we have that γ^* and χ^* lie in the same p -block of \hat{G} . Now, $N \subseteq \mathbf{Z}(\hat{G})$, by Theorem 3.3(a). Thus γ^* and χ^* lie in the same p -block of \hat{G}/N by Lemma 2.2 (a). Hence $\gamma^* \in \text{Irr}(\hat{B}|\hat{\lambda})$, and therefore $\gamma \in B_\theta$. This proves (b). (The part on the defect groups follows from the previous paragraph.)

For part (c), notice that if \mathcal{P} is a representation affording χ , then \mathcal{P} is a projective representation associated with (G, N, θ) with trivial factor set. Hence $\hat{G} = G$ is a representation group for (G, N, θ) with associated character $\tau = \chi$. In this case the standard bijection is the map $\beta\chi \mapsto \beta$ from $\text{Irr}(G|\theta) \rightarrow \text{Irr}(G/N)$ given by Gallagher's Corollary 6.18 of [Is], and part (c) easily follows.

Next we prove part (d). Let $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ be a standard isomorphic character triple and let $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(\hat{G}/N|\hat{\lambda})$ be the standard bijection. Let \hat{B} be the p -block of \hat{G}/N such that $(B_\theta)^* = \text{Irr}(\hat{B}|\hat{\lambda})$. By Theorem 9.2 and Corollary 9.6 of [N] we have that $\text{Irr}(\hat{B}|\hat{\lambda}) = \text{Irr}(\hat{G}/N|\hat{\lambda})$. Therefore $|B_\theta| = |(B_\theta)^*| = |\text{Irr}(\hat{B}|\hat{\lambda})| = |\text{Irr}(\hat{G}/N|\hat{\lambda})| = |\text{Irr}(G|\theta)|$, and the first part of (d) is proved. Let D_θ/N be a θ -defect group of B_θ and let $\hat{D}/N \leq \hat{G}/N$ be a defect group of \hat{B} such that $\pi(\hat{D})/N = D_\theta/N$. Recall that $\hat{\pi} : \hat{G}/N \rightarrow G/N$ defined by $(g, z)N \mapsto gN$ is an onto homomorphism with $\ker(\hat{\pi}) = \hat{N}/N$. Write $G^* = \hat{G}/N$, $N^* = \hat{N}/N$ and $D^* = \hat{D}/N$, and write $\tilde{\pi} : G^*/N^* \rightarrow G/N$ for the induced isomorphism. Then $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$. Let $K^* \in \text{cl}(G^*)$ be a defect class for \hat{B} . By Corollary 3.8 of [N], we know that K^* consists of p -regular elements. Since G^*/N^* is a p -group, we have that $K^* \subseteq N^* \subseteq \mathbf{Z}(G^*)$. Let $x^* \in K^*$ be such that $D^* \in \text{Syl}_p(\mathbf{C}_{G^*}(x^*))$. Since K^* is central, we have that $D^* \in \text{Syl}_p(G^*)$. Then $D^*N^*/N^* \in \text{Syl}_p(G^*/N^*)$ and, since $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$ we have that $D_\theta/N \in \text{Syl}_p(G/N)$. Since G/N is p -group, $D_\theta/N = G/N$. □

If (G, N, θ) is a character triple and $B_\theta \subseteq \text{Irr}(G|\theta)$ is a θ -block, then, in general, B_θ is much smaller than the set $\text{Irr}(B|\theta)$, where B is the Brauer p -block containing B_θ . Indeed, if G is a p -constrained group, for instance, $N \triangleleft G$ is such that p does not divide $|G/N|$, and $\theta \in \text{Irr}(N)$ extends to G , then we have that G has only one p -block B . Thus $\text{Irr}(B|\theta) = \text{Irr}(G|\theta)$, while the θ -blocks have size 1 (by Theorem 5.1(c)).

To end this section, we give an analogue of a classical result on blocks.

Theorem 5.2. *Let $\chi \in \text{Irr}(G|\theta)$ and let B_θ be the θ -block containing χ . Let $g \in G$ and suppose that $(gN)_p$ is not G/N -conjugate to any element of D_θ/N , where D_θ/N is a θ -defect group of B_θ . Then $\chi(g) = 0$.*

Proof. Let $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ be a standard isomorphic character triple of (G, N, θ) . Write $\pi : \hat{G} \rightarrow G$ for the canonical onto homomorphism. Since $\pi(N) = N$, π induces a homomorphism $\hat{\pi} : \hat{G}/N \rightarrow G/N$ with $\ker(\hat{\pi}) = \hat{N}/N$. Write $G^* = \hat{G}/N$ and $N^* = \hat{N}/N$ and write $\tilde{\pi} : G^*/N^* \rightarrow G/N$ for the induced isomorphism.

Let $gN \in G/N$ and let $g^*N^* \in G^*/N^*$ such that $\tilde{\pi}(g^*N^*) = gN$. Let \hat{B} be the p -block of G^* such that $(B_\theta)^* = \text{Irr}(\hat{B}|\hat{\lambda})$, where $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(G^*|\hat{\lambda})$ is the standard bijection. Let $D^* = \hat{D}/N$ be the defect group of \hat{B} such that $\pi(\hat{D})/N = D_\theta/N$. Notice that $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$.

Since $(gN)_p$ is not G/N -conjugate to any element of D_θ/N , we have that $(g^*)_pN^*$ is not G^*/N^* -conjugate to any element of D^*N^*/N^* . Hence $(g^*)_p$ is not contained in any defect group of the block of χ^* . By Corollary 5.9 of [N] we have that $\chi^*(g^*) = 0$. Recall that $\chi^\pi = \tau\chi^*$, where $\tau \in \text{Irr}(\hat{G})$ is the character associated to \mathcal{P} . Since $\tilde{\pi}(((g, 1)N)(\hat{N}/N)) = \hat{\pi}((g, 1)N) = gN$, we have that $g^* = (g, 1)N$ and then

$$\chi(g) = \chi^\pi(g, 1) = \tau(g, 1)\chi^*((g, 1)N) = \tau(g, 1)\chi^*(g^*) = 0.$$

□

6. THEOREM D

In this section we prove Theorem D of the introduction. We will need the following result, which is essentially a result of R. Knörr. Recall that if (G, N, θ) is a character triple, then $xN \in G/N$ is **θ -good** if θ has a D -invariant extension to $N\langle x \rangle$, where $D/N = \mathbf{C}_{G/N}(xN)$. The θ -good conjugacy classes of G/N (those consisting of θ -good elements) play the role of the conjugacy classes of G when we are working with characters of G over θ . For instance, it is a theorem of P. X. Gallagher that $|\text{Irr}(G|\theta)|$ is the number of conjugacy classes of G/N consisting of θ -good elements (see Theorem 5.16 of [N2]).

Theorem 6.1. *Suppose that $Z \subseteq \mathbf{Z}(G)$ and let $\theta \in \text{Irr}(Z)$. Suppose that gZ and hZ are not G/Z -conjugate. Then*

$$\sum_{\chi \in \text{Irr}(G|\theta)} \chi(g)\chi(h^{-1}) = 0.$$

Also

$$\sum_{\chi \in \text{Irr}(G|\theta)} |\chi(g)|^2 = |\mathbf{C}_{G/Z}(gZ)|$$

if g is θ -good.

Proof. The first part is a special case of Corollary 7 of [K]. The second part is an unpublished result of I. M. Isaacs. For a proof see Theorem 5.21 of [N2]. \square

As we said in Section 2, we write $G^{p'}$ for the set of p -regular elements of the finite group G . If $\chi \in \text{Irr}(G)$, we denote by $\chi^{p'}$ the restriction of χ to $G^{p'}$. We know that we can write

$$\chi^{p'} = \sum_{\varphi \in \text{IBr}(G)} d_{\chi\varphi} \varphi$$

for uniquely determined non-negative integers $d_{\chi\varphi}$ called the decomposition numbers. The matrix $D = (d_{\chi\varphi})$ is called the decomposition matrix of G . The following is Theorem D of the introduction.

Theorem 6.2. *Suppose that Z is a central subgroup of G , and let $\theta \in \text{Irr}(Z)$. Let B be a Brauer p -block of G such that $\text{Irr}(B|\theta)$ is not empty. Then the matrix $D_{B,\theta} = (d_{\chi\varphi})$, where $\chi \in \text{Irr}(B|\theta)$ and $\varphi \in \text{IBr}(B)$ is not of the form*

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

Proof. Let $D = (d_{\chi\varphi})$ be the decomposition matrix of G and let M_θ be the submatrix of D whose rows are indexed by the characters in $\text{Irr}(G|\theta) = \{\chi_1, \dots, \chi_k\}$. Let $\{x_1, x_2, \dots, x_l\}$ be a set of representatives of the p -regular conjugacy classes of G . Let $X_\theta = (\chi_i(x_j))$ be the submatrix of the character table of G with rows indexed by elements in $\text{Irr}(G|\theta)$ and columns indexed by the representatives of the p -regular conjugacy classes of G . Let $Y = (\varphi_i(x_j))$ be the Brauer character table of G , where $\text{IBr}(G) = \{\varphi_1, \dots, \varphi_l\}$. Then we have that $X_\theta = M_\theta Y$.

We first assume that Z is a p -group. Suppose that $g \in G$ is p -regular. We claim that g is θ -good. First we prove that $\mathbf{C}_{G/Z}(gZ) = \mathbf{C}_G(g)/Z$. Indeed, let $xZ \in \mathbf{C}_{G/Z}(gZ)$. Then,

$$gZ = (gZ)^{xZ} = g^x Z,$$

and therefore $g^x = gz$ for some $z \in Z$. Since z is a central p -element and g^x and g are p -regular elements, we have that $z = 1$ and therefore $x \in \mathbf{C}_G(g)$. Now let η be an

extension of θ to $\langle Z, g \rangle$. We need to prove that η is $\mathbf{C}_G(g)$ -invariant. But this is clear since $\mathbf{C}_G(g) \subseteq \mathbf{C}_G(x)$ for all $x \in \langle Z, g \rangle$. Hence g is θ -good and the claim is proven.

Note that if x_i and x_j are not G -conjugate p -regular elements, then $x_i Z$ and $x_j Z$ are not G/Z -conjugate. Indeed, suppose that there exists $gZ \in G/Z$ such that $x_i Z = (x_j Z)^{gZ} = x_j^g Z$, hence $x_i = x_j^g z$ for some $z \in Z$. Again, since Z is a central p -group and x_i and x_j^g are p -regular elements, we have that $z = 1$ and hence $x_i = x_j^g$. Let $E \in \text{Mat}_l(\mathbb{C})$ be the diagonal matrix with diagonal entries $|\mathbf{C}_{G/Z}(x_i Z)|$. By Theorem 6.1 we have that

$$E = X_\theta^t \overline{X}_\theta = Y^t (M_\theta)^t M_\theta \overline{Y}.$$

What we have done until now holds for every $\theta \in \text{Irr}(Z)$. If $\theta = 1_Z$ is the trivial character of Z , notice that M_{1_Z} is the decomposition matrix of G/Z , since $\text{Irr}(G|1_Z) = \text{Irr}(G/Z)$ and $\text{IBr}(G/Z) = \text{IBr}(G)$ by Theorem 7.6 of [N]. By the previous equation, for $\theta = 1_Z$, we have

$$E = X_{1_Z}^t \overline{X}_{1_Z} = Y^t (M_{1_Z})^t M_{1_Z} \overline{Y},$$

and since Y is a regular matrix, we conclude that

$$C = M_{1_Z}^t M_{1_Z} = M_\theta^t M_\theta,$$

where C is the Cartan matrix of G/Z . Until now, our ordering of $\text{Irr}(G|\theta) = \{\chi_1, \dots, \chi_k\}$ and $\text{IBr}(G) = \{\varphi_1, \dots, \varphi_l\}$ was arbitrary. Now let B_1, B_2, \dots, B_r be the different p -blocks of G , and order $\text{Irr}(G|\theta)$ and $\text{IBr}(G)$ by blocks (so that the first characters are in B_1 , and so on). Since Z is a central p -group, by Theorem 7.6 of [N] we have that there exists a unique p -block \overline{B}_i of G/Z contained in B_i . Let $C_{\overline{B}_i}$ be the Cartan matrix of \overline{B}_i . We have that $C = \text{diag}(C_{\overline{B}_1}, \dots, C_{\overline{B}_r})$ and $M_\theta = \text{diag}(M_{B_1, \theta}, \dots, M_{B_r, \theta})$ are block diagonal matrices. Then,

$$M_\theta^t M_\theta = \text{diag}(M_{B_1, \theta}^t M_{B_1, \theta}, \dots, M_{B_r, \theta}^t M_{B_r, \theta}).$$

Since $C = M_\theta^t M_\theta$, we necessarily have that $C_{\overline{B}_i} = M_{B_i, \theta}^t M_{B_i, \theta}$ for every i . Now if $M_{B, \theta}$ is of the form

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix},$$

so is $C_{\overline{B}}$. By Problem 3.4 of [N] this is a contradiction.

This ends the proof of the case where Z is a p -group. We prove now the general case. Write $Z = Z_p \times Z_{p'}$, where Z_p is the Sylow p -subgroup of Z and write $\theta = \theta_p \times \theta_{p'}$, with $\theta_p \in \text{Irr}(Z_p)$ and $\theta_{p'} \in \text{Irr}(Z_{p'})$. By assumption, there is $\chi \in \text{Irr}(B)$ over θ , and therefore over $\theta_{p'}$. Now, B covers the block $\{\theta_{p'}\}$, and by Theorem 9.2 of [N], we have that $\text{Irr}(B|\theta_{p'}) = \text{Irr}(B)$. We conclude that $D_{B, \theta} = D_{B, \theta_p}$, and we are done by the central p -group case. \square

Notice, for instance, that if $G = A_4$, $p = 2$, $Z \triangleleft G$ is the Klein subgroup, and $\theta = 1_Z$, then G has a unique 2-block, and the matrix $D_{B,\theta}$ is the identity.

7. THEOREMS B AND C

We start this section by proving Theorem C of the introduction. Recall that Brauer's $k(B)$ -conjecture asserts that if B is a block with defect group D , then $k(B) = |\text{Irr}(B)| \leq |D|$. The key is to use Theorem C of [N1].

Theorem 7.1. *The $k(B)$ -conjecture is true for every finite group if and only if for every character triple (G, N, θ) , we have that every θ -block B_θ has size less than or equal the size of any of its θ -defect groups.*

Proof. Let (G, N, θ) be a character triple and let $(\hat{G}/N, \hat{N}/N, \hat{\lambda})$ be a standard isomorphic character triple. Write $G^* = \hat{G}/N$, $N^* = \hat{N}/N$ and $\theta^* = \hat{\lambda}$. Let B_θ be a θ -block and let D_θ/N be a θ -defect group of B_θ . Suppose first that the $k(B)$ -conjecture holds for every finite group. Write $N^* = N_p^* \times N_{p'}^*$, where $N_p^* \in \text{Syl}_p(N^*)$, and write $\theta^* = \theta_p^* \times \theta_{p'}^*$, with $\theta_p^* \in \text{Irr}(N_p^*)$ and $\theta_{p'}^* \in \text{Irr}(N_{p'}^*)$. From the definition of the θ -blocks, we have that there exists a p -block B^* of G^* such that $B_\theta^* = \text{Irr}(B^*|\theta^*)$, where $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(G^*|\theta^*)$ is the standard bijection. Thus

$$|B_\theta| = |\text{Irr}(B^*|\theta^*)|.$$

Now by Theorem 9.2 of [N], and using that $\text{Irr}(B^*|\theta^*) \neq \emptyset$ (by the definition of θ -blocks), we have that $\text{Irr}(B^*|\theta^*) = \text{Irr}(\overline{B^*})$. By Theorem C of [N1] we have that $|\text{Irr}(B^*|\theta_p^*)| \leq |\text{Irr}(\overline{B^*})|$, where $\overline{B^*}$ is the unique p -block of G^*/N_p^* contained in B^* . Let D^* be a defect group of B^* . By Theorem 9.10 of [N] we have that D^*/N_p^* is a defect group of $\overline{B^*}$. Since the $k(B)$ conjecture holds for G^*/N_p^* we have that $|\text{Irr}(\overline{B^*})| \leq |D^*/N_p^*| = |D^*N^*/N^*|$. Now, write $D^* = \hat{D}/N$, for some subgroup \hat{D} of \hat{G} , and by definition, recall that $\pi(\hat{D})/N = D_\theta/N$ is a θ -defect group of B_θ , where $\pi : \hat{G} \rightarrow G$ is the onto homomorphism $(g, z) \mapsto g$. It is then enough to show that $|D^*N^*/N^*| = |D_\theta/N|$. Notice that $\hat{\pi} : G^* \rightarrow G/N$ defined by $(g, z)N \mapsto gN$ is an onto group homomorphism with kernel N^* . Write $\tilde{\pi} : G^*/N^* \rightarrow G/N$ for the isomorphism induced by $\hat{\pi}$, and notice that $\tilde{\pi}(D^*N^*/N^*) = \hat{\pi}(D^*) = \pi(\hat{D})/N = D_\theta/N$. Then D_θ/N and D^*N^*/N^* are isomorphic, and $|D_\theta/N| = |D^*N^*/N^*|$, as desired.

For the converse, simply take $N = 1$ and apply Theorem 5.1(c). □

Next we prove Theorem B of the introduction. The key is Theorem 3 of [S]. Recall that Conjecture A asserts the following: if $B_\theta \subseteq \text{Irr}(G|\theta)$ is a θ -block with θ -defect group D_θ/N and θ extends to D_θ , then $(\chi(1)/\theta(1))_p = |G : D_\theta|_p$ for all $\chi \in B_\theta$ if and only if D_θ/N is abelian.

Theorem 7.2. *Conjecture A and Brauer's Height Zero conjecture are equivalent.*

Proof. Let (G, N, θ) be a character triple, and let B_θ be a θ -block with θ -defect group D_θ/N . As in the proof of Theorem 7.1, let (G^*, N^*, θ^*) be a standard isomorphic triple, with standard bijection $*$: $\text{Irr}(G|\theta) \rightarrow \text{Irr}(G^*|\theta^*)$, and suppose that B^* is the block of G^* such that $(B_\theta)^* = \text{Irr}(B^*|\theta^*)$. Recall that $N^* \subseteq \mathbf{Z}(G^*)$. We have shown above that D_θ/N is isomorphic to D^*N^*/N^* , where D^* is a defect group of B^* . (In fact, we have shown that if $\tilde{\pi} : G^*/N^* \rightarrow G/N$ is the group isomorphism induced by $\pi : \hat{G} \rightarrow G$, then $\tilde{\pi}(D^*N^*/N^*) = D_\theta/N$.) Notice that, since N^* is central in G^* , we have that the Sylow p -subgroup of N^* is contained in D^* (Theorem 4.8 of [N]), and therefore $|D^*N^* : D^*|_p = 1$. Then

$$|G : D_\theta|_p = |G/N : D_\theta/N|_p = |G^* : D^*N^*|_p = |G^* : D^*|_p.$$

Now, as is well-known, character triple isomorphisms preserve ratios of character degrees (see Lemma 11.24 of [Is]), that is $\chi(1)/\theta(1) = \chi^*(1)/\theta^*(1) = \chi^*(1)$ for $\chi \in \text{Irr}(G|\theta)$. In particular, if $\chi \in B_\theta$, then

$$(\chi(1)/\theta(1))_p = \chi^*(1)_p = |G^* : D^*|_p p^{h(\chi^*)} = |G : D_\theta|_p p^{h(\chi^*)},$$

where $0 \leq h(\chi^*)$ is the height of χ^* in B^* .

By the properties of character triple isomorphisms, notice that θ extends to D_θ if and only if θ^* extends to D^*N^* . Now, it is clear that Conjecture A and the projective version of the Height Zero conjecture due to G. Malle and G. Navarro (Conjecture A of [MN]) are equivalent. By Theorem 3 of [S], we are done. \square

We end this paper with the following observation.

Proposition 7.3. *Conjecture A implies the Gluck-Wolf-Navarro-Tiep theorem.*

Proof. Suppose that p does not divide $\chi(1)/\theta(1)$ for all $\chi \in \text{Irr}(G|\theta)$. Let B_θ be a θ -block and let $\chi \in B_\theta$. Since $(\chi(1)/\theta(1))_p = |G : D_\theta|_p p^{h(\chi^*)}$, we have that $|G : D_\theta|$ is not divisible by p . Hence D_θ/N is a Sylow p -subgroup of G/N . Since p does not divide $\chi(1)/\theta(1)$, and all the irreducible constituents of χ_{D_θ} lie over θ , it follows that there is some irreducible constituent $\gamma \in \text{Irr}(D_\theta|\theta)$ such that p does not divide $\gamma(1)/\theta(1)$. By Corollary 11.29 of [Is], we have that $\gamma_N = \theta$. Since we are assuming that Conjecture A holds, we have that D_θ/N is abelian. Since $D_\theta/N \in \text{Syl}_p(G/N)$, we have that G/N has abelian Sylow p -subgroups. \square

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DEPARTAMENT DE MATEMÀTIQUES, UNIVERSITAT DE VALÈNCIA, 46100 BURJASSOT, VALÈNCIA, SPAIN

E-mail address: noelia.rizo@uv.es